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ON THE LATTICE OF SUBALGEBRAS OF AN ALGEBRA[1]

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ABSTRACT. Let R be a Noetherian inertial coefficient ring and let A be a finitely generated R-algebra (that is, finitely generated as an R-module) with Jacobson radical $J(A)$. Let S be a subalgebra of A with $S + J(A) = A$. We show that for every separable subalgebra T of A there is a unit a of A such that $aTa^{-1} \subseteq S$. It follows that if S is separable (hence inertial) and if T is a maximal separable subalgebra of A, then T is inertial. We also show that if $S + I = A$ for a nil ideal I of A, then R can be taken to be an arbitrary commutative ring, and the conjugacy result still holds.

All rings will be associative and will possess an identity element 1. All subrings will contain the identity of the overring, and ring homomorphisms will map the identity to the identity.

If A is a ring, R a commutative ring, and θ a ring homomorphism of R into the center of A, then θ induces a natural R-module structure on A defined by $r \cdot a = \theta(r)a$ for $r \in R$, $a \in A$, and we say that A is an *R-algebra*. An R-algebra A is said to be *finitely generated* or *projective* if it is finitely generated or projective as a module over R. For all rings R we let $J(R)$ denote the Jacobson radical (or radical) of R. We denote by μ the multiplication map μ: $A \otimes A^{\circ} \to A$ and let $J = \ker \mu$.

Recall that an R-algebra A is *separable over R* (or separable) if its enveloping algebra $A \otimes_R A^{\circ}$ contains an idempotent e with the property that $\mu(e) = 1$ and $J \cdot e = 0$. If A is a finitely generated algebra over a commutative ring R, then a subalgebra S of A is an *inertial subalgebra* if S is a separable R-algebra such that $S + J(A) = A$. A commutative ring R is an *inertial coefficient ring* if every finitely generated R-algebra A for which $A/J(A)$ is separable contains an inertial subalgebra. Basic properties of separable algebras can be found in [3], and basic properties of inertial subalgebras and inertial coefficient rings can be found in [7].

If A is a finitely generated R-algebra and I is an ideal of A, then we say we can "lift idempotents from A/I to A" if every idempotent in A/I is the image of an idempotent in A under the natural map from A to A/I. In [9, Theorem 4, p. 221] Kirkman proved that if R is an inertial coefficient ring and A a finitely generated R-algebra, then idempotents can be lifted from $A/J(A)$ to A. Ingraham has

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conjectured that the converse is true as well; that is, if R has the property that idempotents can be lifted from $A/J(A)$ to A for every finitely generated R-algebra A, then R is an inertial coefficient ring.

Let R be a commutative ring and A a finitely generated A-algebra. We are interested in finding conditions under which a maximal separable subalgebra T of A is inertial. It is clear that if S is an inertial subalgebra of A, and if a is a unit of A such that $aTa^{-1} \subseteq S$, then T is inertial. Thus we are led to look for conditions under which we can conjugate T into S.

We consider the more general question: Under what circumstances can any separable subalgebra T of A be conjugated into any subalgebra S with $S + J(A) = A$? There are examples due to N. Ford [**4**] of rings R and finitely generated R-algebras A containing nonisomorphic inertial subalgebras. Since inertial subalgebras are maximal separable subalgebras [**7**, Theroem 2.5, p. 80], such conjugation does not always occur. The main result of this paper is to prove that it does occur whenever R is a Noetherian inertial coefficient ring.

THEOREM 1.1. *Let R be a Noetherian inertial coefficient ring, and let A be a finitely generated R-algebra. Let T be a separable subalgebra of A, and let S be a subalgebra of A with the property that $S + J(A) = A$. Then there is a unit a of A such that $aTa^{-1} \subseteq S$.*

PROOF. *Step* 1. We first reduce to the case where $A/J(A)$ is separable over R. Let $A_1 = T + J(A)$. Then $J(A) \subseteq J(A_1)$ [**1**, Corollary, p. 126], so $A_1/J(A_1)$ is a homomorphic image of T and hence is separable. Setting $S_1 = S \cap A_1$, we clearly have $S_1 + J(A) \subseteq A_1$. To show equality, we write an arbitrary element a_1 of $A_1 = T + J(A)$ as $t + n$ where $t \in T$, $n \in J(A)$. Since a_1 also lies in $A = S + J(A)$, we have $t + n = s + n_1$ for $s \in S$, $n_1 \in J(A)$. It follows that $s = t + n - n_1$ is in $S \cap (T + J(A)) = S_1$, so that $a_1 \in S_1 + J(A)$, and we have $S_1 + J(A) = A_1$. If the theorem is true for A_1, then there is a unit $a \in A_1 \subseteq A$ such that $aTa^{-1} \subseteq S_1 \subseteq S$. Thus, it suffices to prove the theorem in the case that $A/J(A)$ is separable over R.

Step 2. We now reduce to the case where S is separable, hence inertial. By [**1**, Corollary, p. 126], $S \cap J(A) \subseteq J(S)$, but $S/[S \cap J(A)] \simeq A/J(A)$ is semisimple, so $S \cap J(A) = J(S)$. Since R is an inertial coefficient ring, S contains a separable subalgebra S_1 such that $S_1 + J(S) = S$, and it follows that $S_1 + J(A) = A$. Clearly, if we can conjugate T into S_1, we can conjugate it into S. Therefore, we assume S is an inertial subalgebra.

The remainder of the proof involves the following setting. Let $\bar{A} = A/J(A)$, $\bar{T} = T/(T \cap J(A))$, and $\bar{R} = R/(R \cap J(A))$. Let $f\colon S \otimes_R T^\circ \to \bar{A} \otimes_{\bar{R}} \bar{T}^\circ$ and $g\colon T \otimes_R T^\circ \to \bar{A} \otimes_{\bar{R}} \bar{T}^\circ$ be the natural maps, and let e be a separability idempotent for T with $\bar{e} = g(e)$. Then

$$\ker f = i[(S \cap J(A)) \otimes_R T^\circ + S \otimes_R (T \cap J(A))^\circ] \subseteq J(S \otimes_R T^\circ)$$

[**1**, Theorem 10, p. 127]. Since R is an inertial coefficient ring, idempotents can be lifted from $(S \otimes_R T^\circ)/J(S \otimes_R T^\circ)$ to $S \otimes_R T^\circ$, and it follows from [**6**, Corollary 1.3, p. 46] that we can lift idempotents from $\bar{A} \otimes_{\bar{R}} \bar{T}^\circ$ to $S \otimes_R T^\circ$, so let e_1 be an idempotent in $S \otimes_R T^\circ$ such that $f(e_1) = \bar{e}$. The picture looks like this:

$$\begin{tikzcd}
S \otimes_R T^\circ \arrow[rr, "f"] & & \bar{A} \otimes_{\bar{R}} \bar{T}^\circ \\
e_1 & & g(e) = \bar{e} = f(e_1) \\
& T \otimes_R T^\circ \arrow[uur, "g"'] & \\
& e &
\end{tikzcd}$$

If μ: $A \otimes_R A^\circ \to A$ is the multiplication map, then we will show that $\mu(e_1)$ is the conjugating element we seek. In other words, providing e_1 is an *idempotent* preimage of $\bar{e}$, we will show that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

Step 3. Let R be a field. The proof of [**2**, Lemma 2.7, p. 127] gives the existence of a unit α in A of the form $\alpha = 1 + n$ for n in $J(A)$ such that $\alpha T \alpha^{-1} \subseteq S$. If we define φ: $T \to S$ by $\varphi(t) = \alpha t \alpha^{-1}$, then the map $\varphi \otimes 1$: $T \otimes_R T^\circ \to S \otimes_R T^\circ$ makes the preceding diagram commute. Furthermore,

$$\ker f = i[(S \cap J(A)) \otimes_R T^\circ + S \otimes_R (T \cap J(A))^\circ] = 0,$$

since S and T are separable over the field R. Therefore, if we let $e_1 = (\varphi \otimes 1)(e)$, then e_1 is the unique preimage of $\bar{e}$ in $S \otimes_R T^\circ$, and e_1 is also an idempotent.

Because e is a separability idempotent for T, we have $(1 \otimes t - t \otimes 1) \cdot e = 0$ for every t in T. Applying $\varphi \otimes 1$, we have $(1 \otimes t - \varphi(t) \otimes 1) \cdot e_1 = 0$. Next apply μ, recall that $\varphi(t) = \alpha t \alpha^{-1}$, and notice that $\mu(e_1)t - \alpha t \alpha^{-1}\mu(e_1) = 0$. It follows that $\mu(e_1)t\mu(e_1)^{-1} = \alpha t \alpha^{-1}$ is in S, provided $\mu(e_1)$ is invertible. But $\mu(\bar{e}) = 1$, so $\mu(e_1) = 1 + n$ for some $n \in J(A)$; consequently $\mu(e_1)$ is invertible.

Step 4. Suppose (R, m) is a Noetherian local ring with $m^n = 0$ for some positive integer n. We proceed by induction on n. If $n = 1$, then R is a field, and the result follows from Step 3. Assume the statement is true for $n \leqslant k$, and consider the case where $n = k + 1$.

Let $\tilde{A} = A/(m^k A)$, $\tilde{R} = R/m^k$, $\tilde{T}/(m^k A \cap T)$, and $\tilde{S} = S/(m^k A \cap S)$. Since $m^k A \subseteq J(A)$ by [**7**, Lemma 1.1, p. 78], then $J(\tilde{A}) = J(A)/m^k A$. Letting $\tilde{e}_1$ and $\tilde{e}$ be the images of e_1 and e, and taking $\tilde{f}$ and $\tilde{g}$ to be the induced maps from f and g, we have the following situation:

$$\begin{tikzcd}
\tilde{S} \otimes_{\tilde{R}} \tilde{T}^\circ \arrow[rr, "\tilde{f}"] & & \bar{A} \otimes_{\bar{R}} \bar{T}^\circ \\
\tilde{e}_1 & & \tilde{g}(\tilde{e}) = \bar{e} = \tilde{f}(\tilde{e}_1) \\
& \tilde{T} \otimes_{\tilde{R}} \tilde{T}^\circ \arrow[uur] & \\
& \tilde{e} &
\end{tikzcd}$$

Both $\tilde{T}$ and $\tilde{S}$ are separable over $\tilde{R}$, $\tilde{e}$ is a separability idempotent for $\tilde{T}$, and $\tilde{S} + J(\tilde{A}) = \tilde{A}$. Then the induction hypothesis gives that $\mu(\tilde{e}_1)\tilde{T}\mu(\tilde{e}_1)^{-1} \subseteq \tilde{S}$. Pulling this inclusion back to A, we have $\mu(e_1)T\mu(e_1)^{-1} \subseteq S + m^k A$.

Now let $C = S + m^k A$ and $T' = \mu(e_1)T\mu(e_1)^{-1}$. S is an inertial subalgebra of C, and C is a finitely generated R-algebra because R is Noetherian. Write $e = \sum \gamma_i \otimes \delta_i$, where $\gamma_i \in T$, $\delta_i \in T^o$, and let

$$e' = \sum \left[\mu(e_1)\gamma_i\mu(e_1)^{-1} \otimes \mu(e_1)\delta_i\mu(e_1)^{-1}\right].$$

One easily sees that e' is a separability idempotent for T'. Write $e_1 = \sum \alpha_j \otimes \beta_j$ where $\alpha_j \in S$ and $\beta_j \in T^o$, let $e_1' = \sum \alpha_j \otimes \mu(e_1)\beta_j\mu(e_1)^{-1}$, and notice that e_1' is an idempotent. It is not hard to see that $J(C) = C \cap J(A)$, $C/J(C) = \bar{A}$, and $T'/(T' \cap J(C)) \simeq \bar{T}$. Thus, we have natural maps f': $S \otimes_R T'^o \to \bar{A} \otimes_{\bar{R}} \bar{T}^o$ and g': $T' \otimes_R T'^o \to \bar{A} \otimes_{\bar{R}} \bar{T}^o$ with $f'(e'_1) = \bar{e} = g'(e')$. We can now use the same argument here for C that we used previously for A to conclude that

$$\mu(e_1')T'\mu(e_1')^{-1} \subseteq S + m^kC = S + m^k(S + m^kA) = S.$$

Equivalently, $\mu(e_1')\mu(e_1)T\mu(e_1)^{-1}\mu(e_1')^{-1} \subseteq S$. But

$$\begin{aligned}\mu(e_1')\mu(e_1) &= \left(\sum \alpha_j\mu(e_1)\beta_j\mu(e_1)^{-1}\right) \cdot \mu(e_1) = \sum \alpha_j\mu(e_1)\beta_j \\ &= \left[\sum \alpha_j \otimes \beta_j\right] \cdot \mu(e_1) = e_1 \cdot \mu(e_1) = \mu(e_1 \cdot e_1) = \mu(e_1).\end{aligned}$$

Thus we have shown that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

Step 5. Let (R, m) be a Noetherian local ring. Let k be a positive integer, and pass to the factor algebra $\tilde{A} = A/m^kA$ over $\tilde{R} = R/m^k$. Letting $\tilde{T} = T/(m^kA \cap T)$ and $\tilde{S} = S/(m^kA \cap S)$, we have that $\tilde{T}$ is $\tilde{R}$-separable and $\tilde{S}$ is an $\tilde{R}$-inertial subalgebra of $\tilde{A}$. Defining $\tilde{e}$, $\tilde{e}_1$, $\tilde{f}$, and $\tilde{g}$ as in Step 4, we apply the result of Step 4 to get $\mu(\tilde{e}_1)\tilde{T}\mu(\tilde{e}_1)^{-1} \subseteq \tilde{S}$. Pulling back to A we have $\mu(e_1)T\mu(e_1)^{-1} \subseteq S + m^kA$. The containment holds for every positive integer k, so $\mu(e_1)T\mu(e_1)^{-1} \subseteq \bigcap_{k=1}^{\infty}(S + m^kA)$. But R is a Zariski ring [**11**, pp. 263–264], so $\bigcap_{k=1}^{\infty}(S + m^kA) = S$, and again we have shown that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

Step 6. Let R be a Noetherian ring and $T' = \mu(e_1)T\mu(e_1)^{-1}$. We will show that $T' \subseteq S$ by showing that $Z = (T' + S)/S$ is the zero module. $Z = 0$ if and only if $Z_m = Z \otimes_R R_m = 0$ for every maximal ideal m of R [**3**, Proposition 4.4, p. 29]. By tensoring everything in the diagram in Step 2 with R_m over R, we again place ourselves in the setting of Step 5, where we have $T'_m \subseteq S_m$, or equivalently, $Z_m = 0$. We conclude that $Z = 0$, and it follows that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$. $\square$

EXAMPLE. Let $R = Z/4Z$, $A = M_2[R[x]/(x^2 - 2)]$, and $S = M_2(R)$. Then $J(A) = A\left[\begin{smallmatrix}\bar{x} & 0\\ 0 & \bar{x}\end{smallmatrix}\right] + 2A$, so S is an inertial subalgebra of A. Moreover,

$$T = \left\{\begin{bmatrix} 3a + 2b & (a + 3b)\bar{x} \\ (3a + b)\bar{x} & 2a + 3b \end{bmatrix} a, b \in R\right\}$$

is a separable subalgebra of A with separability idempotent

$$e = \begin{bmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{bmatrix} \otimes \begin{bmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{bmatrix}.$$

Let

$$z = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{bmatrix},$$

where $z \in S \otimes_R T^\circ$ and $f(z) = \bar{e} = g(e)$. Notice that z is not an idempotent, and furthermore, that $\mu(z)T\mu(z)^{-1} \not\subseteq S$. Thus we see that there are nonidempotent preimages z of $\bar{e}$ in $S \otimes_R T^\circ$ for which $\mu(z)$ does not conjugate T into S.

In the proof of Theorem 1.1, once the reductions of Steps 1 and 2 are made, the only place we use that R is an inertial coefficient ring is when we wish to lift idempotents. Thus, if we start with the assumptions that $A/J(A)$ is separable and S is an inertial subalgebra of A, we have the following corollary.

COROLLARY 1.2. *Let R be a Noetherian ring with the property that for every finitely generated R-algebra, idempotents can be lifted from the algebra modulo its radical to the algebra. Let A be a finitely generated R-algebra with $A/J(A)$ separable, let S be an inertial subalgebra of A, and let T be a separable subalgebra of A. Then there is a unit a in A with $aTa^{-1} \subseteq S$.*

Conjugates of inertial subalgebras are inertial subalgebras, so if we are in a setting where inertial subalgebras exist and Theorem 1.1 applies, the following corollary holds.

COROLLARY 1.3. *If R is a Noetherian inertial coefficient ring and A is a finitely generated R-algebra with $A/J(A)$ separable, then every separable subalgebra is contained in an inertial subalgebra, and every maximal separable subalgebra is an inertial subalgebra.*

When A is a commutative, finitely generated algebra over a commutative ring R, the situation becomes much simpler.

PROPOSITION 1.4. *Let A be a commutative, finitely generated algebra over a commutative ring R. Let S be a subalgebra of A with $S + J(A) = A$. If T is a separable subalgebra of A, then $T \subseteq S$.*

PROOF. If we consider A as an S-algebra, then S is an S-inertial subalgebra of A. Then $S \otimes_R T$ is S-separable, and $S \cdot T$ is a homomorphic image of $S \otimes_R T$, so $S \cdot T$ is an S-separable subalgebra of A. Furthermore, since $S \subseteq S \cdot T$, $S \cdot T$ is also an S-inertial subalgebra of A. Therefore, by [7, Proposition 2.6, p. 80], $S \cdot T = S$, and consequently $T \subseteq S$. □

The following result removes both the Noetherian and inertial coefficient ring conditions on R in Theorem 1.1, but we are forced to replace the Jacobson radical of A with a nil ideal I of A. Ford's example shows that we cannot expect this result to be true for $J(A)$ instead of I. It is still unknown whether Theorem 1.1 is true when R is an inertial coefficient ring which is not necessarily Noetherian.

PROPOSITION 1.5. *Let R be a commutative ring and A be a finitely generated R-algebra. Let I be any nil ideal of A, and let S be an R-subalgebra of A such that $S + I = A$. If T is any separable subalgebra of A, then there exists an element a in A such that $aTa^{-1} \subseteq S$.*

PROOF. We use the technique of selecting a suitable Hilbert subring R_1 of R and an R_1-algebra A_1 which satisfy the conditions of Theorem 1.1. We then lift the result back to A.

By [**10**, Theorem 5, p. 5], T is a finitely generated R-algebra, so write $T = Rt_1 + Rt_2 + \cdots + Rt_m$ for $t_i \in T$. T is R-separable so there are elements x_i and y_i in T such that $\sum x_i \otimes y_i$ is a separability idempotent for T in $T \otimes_R T^\circ$. Thus we have, for every $j = 1,\ldots,m$,

$$(*) \qquad (t_j \otimes 1 - 1 \otimes t_j)\left(\sum x_i \otimes y_i\right) = 0 \quad \text{in } T \otimes_R T^\circ.$$

Think of $T \otimes_R T^\circ$ as a free abelian group with subgroup $\mathcal{Q}$ of relations factored out, and notice that there is a finite subset M_j of $T \cup R$ such that the elements of $\mathcal{Q}$ making $(*)$ zero in $T \otimes_R T^\circ$ are expressible in terms of the elements of M_j.

Let $a_1,\ldots,a_n$ generate A as an R-module. Since $S + I = A$, there exist $s_1,\ldots,s_n$ in S and $\mu_1,\ldots,\mu_n$ in I with $a_i = s_i + \mu_i$ for $i = 1,\ldots,n$.

Now set $B = \{1, a_ia_j, s_i, t_i\}$ and $C = \{1, t_it_j, x_i, y_i\} \cap (\bigcup_j M_j)$. Write each element of the finite set B as an R-linear combination of $a_1,\ldots,a_n$, and write each element of the finite set C as an R-linear combination of $t_1,\ldots,t_m$. All of this will involve only finitely many coefficients from R. Let R_1 be the Noetherian subring of R generated by this finite set and the "prime" subring P of R. P is a homomorphic image of the Hilbert ring Z, the integers, so P is a Hilbert ring. R_1 is finitely generated as an algebra over P, so R_1 is a Hilbert ring [**5**, Theorems 2 and 3, pp. 136–137]. Therefore, R_1 is an inertial coefficient ring [**8**, Corollary 2, p. 553].

Define $A_1 = R_1a_1 + R_1a_2 + \cdots + R_1a_n$. By construction of R_1, we have $B \subseteq A_1$, so A_1 is a finitely generated R_1-algebra containing the s_i's and the t_i's. Consequently, we can take S_1 to be the R_1-subalgebra of A_1 generated by $s_1,\ldots,s_n$. Next let $T_1 = R_1t_1 + R_1t_2 + \cdots + R_1t_m$, so T_1 is a finitely generated R_1-algebra containing the set C. Furthermore, $\sum x_i \otimes y_i$ is an element of $T_1 \otimes_{R_1} T_1^\circ$ satisfying $(*)$ in $T_1 \otimes_{R_1} T_1^\circ$. Consequently, $\sum x_i \otimes y_i$ is a separability idempotent for T_1, so T_1 is R_1-separable. Finally, we let $I_1 = I \cap A_1$. Since I is nil, I_1 is nil, and it follows that $I_1 \subseteq J(A_1)$. Recall that $a_i - s_i = \mu_i$ is in I, and since $a_i - s_i$ is also in A_1, then $a_i - s_i = \mu_i$ is in I_1 for $i = 1,2,\ldots,n$. The relations $a_i = s_i + \mu_i$ imply that $S_1 + I_1 = A_1$, and it follows that $S_1 + J(A_1) = A_1$.

Now A_1 satisfies all the conditions of Theorem 1.1, so there is a unit a in A_1 such that $aT_1a^{-1} \subseteq S_1$. We extend back up to A by multiplying by R to get $RA_1 = A$, $RS_1 \subseteq S$, and $RT_1 = T$. Consider a now as an element of A, we have $aTa^{-1} = a(RT_1)a^{-1} = R(aT_1a^{-1}) \subseteq RS_1 \subseteq S$, and we are done. $\square$

Note. If R is a Hilbert ring, the hypothesis of Proposition 1.5 are fulfilled for $I = J(A)$.

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