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ON IDEAL CLASS GROUPS OF 2-POWER EXPONENT

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ABSTRACT. It is shown that for a fixed totally real algebraic number field k and a fixed positive integer t, there exist only finitely many totally imaginary quadratic extensions K of k having ideal class groups of exponent 2^t.

Under the assumption of the extended Riemann Hypothesis it has been shown by Boyd and Kisilevsky **[1]**, and independently by Weinberger **[6]**, that the exponent of the ideal class group of an imaginary quadratic field $\mathbf{Q}(\sqrt{d})$, d a fundamental discriminant, tends to infinity as $|d|$ tends to infinity. It follows that for any positive integer e there exist only finitely many such fields having ideal class group of exponent e. The particular cases $e = 2$, 3 and 4 of this result have been established without the Riemann Hypothesis assumption in **[2, 1** and **3]**, respectively.

The purpose of this note is to prove an unconditional result of this nature for an infinite family of exponents in the more general context of totally imaginary quadratic extensions of an arbitrary totally real algebraic number field.

THEOREM 1. *For a fixed totally real algebraic number field k and a fixed positive integer t, there exist only finitely many totally imaginary quadratic extensions K of k having ideal class groups of exponent 2^t.*

To facilitate the discussion here, we first fix some notations. For an algebraic number field F denote by R_F the ring of algebraic integers of F, by I_F the group of fractional R_F-ideals of F, and by P_F the subgroup of I_F consisting of principal ideals. C_F will denote the ideal class group I_F/P_F of F, whose order will be denoted by h_F, and U_F will denote the group of units of R_F. For an ideal J in I_F, $N(J)$ will denote the absolute norm.

Throughout the paper the fields K and k will be as in Theorem 1. The nontrivial k-automorphism of K will be denoted by *, and the relative discriminant of K over k by $d_{K/k}$. For a fixed positive integer t, let

$$G_t = \left\{x \in C_K : x^{2^t} = 1\right\}.$$

We will prove the following result, from which Theorem 1 follows immediately:

THEOREM 2. *Let k be a fixed totally real algebraic number field and t a fixed positive integer. For totally imaginary quadratic extensions K of k,*

$$|C_K/G_t| \to \infty \quad \textit{as } N(d_{K/k}) \to \infty.$$

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PROOF. Observe first that $C_K/G_t \cong C_K^{2^t}$, and

$$|C_K^{2^t}| = |C_K| \left[\prod_{j=0}^{t-1} |C_K^{2^j}/C_K^{2^{j+1}}| \right]^{-1}.$$

To estimate the factors in the above product it is convenient to introduce the groups

$$H_i = \left\{ B^{2^i} E^{2^{i-1}} P_K \in C_K : B \in I_K \text{ and } E \in I_k \right\}.$$

Consider the mappings $\phi_j : I_K \to C_K^{2^j}/H_{j+1}$ given by $X \mapsto X^{2^j} P_K H_{j+1}$. If $X = aY/Y^*$ for $a \in \dot{K}$, $Y \in I_K$, then

$$X^{2^j} = a^{2^j} Y^{2^j} / (Y^*)^{2^j} = a^{2^j} Y^{2^{j+1}} / (nY)^{2^j},$$

where nY is the relative norm of the ideal Y in the extension K/k. Thus, S is contained in the kernel of ϕ_j, where

$$S = \left\{ aY/Y^* : a \in \dot{K},\ Y \in I_K \right\}.$$

So, for each j, we have

$$|C_K^{2^j}/H_{j+1}| = |I_K/\ker(\phi_j)| \leqslant [I_K : S].$$

The latter index is known from the theory of quadratic extensions of algebraic number fields (see [4, §13, Satz 13]);

$$[I_K : S] = h_k \left[N_{K/k}(K) \cap U_k : U_k^2 \right] 2^{r-1},$$

where r is the number of (finite) primes of k which are ramified in K. For each j,

$$\begin{aligned} |C_K^{2^j}/C_K^{2^{j+1}}| &= |C_K^{2^j}/H_{j+1}|\,|H_{j+1}/C_K^{2^{j+1}}| \\ &\leqslant h_k \left[U_k : U_k^2 \right] 2^{r-1} \cdot h_k = c_1 2^{r-1} \end{aligned}$$

for a constant c_1 depending only upon k. It now follows that

$$|C_K^{2^t}| \geqslant c_2 h_K / (2^t)^{r-1}$$

for a constant c_2 depending upon k and t. Applying a result of Stark [5, Theorem 2] when $k \neq \mathbf{Q}$ and the Brauer-Siegel Theorem when $k = \mathbf{Q}$, h_K can be bounded from below by

$$h_K \geqslant c_3 N(d_{K/k})^{1/4}$$

with c_3 a constant depending only upon k. Combining the last two inequalities, we obtain

$$|C_K^{2^t}| \geqslant c_4 N(d_{K/k})^{1/4} (2^t)^{1-r},$$

where c_4 is a constant depending upon k and t.

Now let m be the number of (not necessarily distinct) prime factors of $N(d_{K/k})$, and let $n = [k : \mathbf{Q}]$. Then

$$N(d_{K/k})^{1/4} (2^t)^{1-r} \geqslant \left(\prod p^{1/4} \right) (2^t)^{-mn} = \prod \left(p^{1/4} 2^{-tn} \right),$$

where the products are taken over the (not necessarily distinct) primes p dividing $N(d_{K/k})$. As $N(d_{K/k})$ tends to infinity, the size of the largest prime divisor of $N(d_{K/k})$ tends to infinity. Hence, the above product tends to infinity and the proof is complete.

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