PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

ON DOUBLE CENTRALIZER SUBGROUPS OF SOME FINITE p-GROUPS

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ABSTRACT. *Let A be a maximal abelian normal subgroup of a finite p-group G ($p > 2$) such that $[G, A]$ is cyclic. Then* (i) $C_G(C_G(D)) = D$ *and* $[G : C_G(D)] = [D : Z(G)]$ *for every* $Z(G) \leqslant D \leqslant G$; (ii) $[G : Z(G)] = [G, A]^2$ *and every faithful absolutely irreducible representation of G is of degree* $[G : A]$. *The case* $p = 2$ *will also be mentioned.*

1. Introduction. Let p be a prime number. For a finite p-group Q, we write $\Omega(Q)$ for $\Omega_2(Q)$ if $p = 2$ and $\Omega_1(Q)$ if $p > 2$, where $\Omega_i(Q) = \langle x \in Q \mid x^{p^i} = 1 \rangle$. Beside this, the notation is standard (cf. [**3** or **5**]).

The main result of this paper is

THEOREM A. *Let A be a maximal abelian normal subgroup of a finite p-group G. Suppose $[G, A]$ is cyclic and $\Omega([G, A]) \subseteq Z(G)$. Then*

(i) $C_G(C_G(D)) = D$ *and* $[G : C_G(D)] = [D : Z(G)]$ *for every* $Z(G) \leqslant D \leqslant A$;

(ii) $[G : Z(G)] = [G, A]^2$ *and every faithful absolutely irreducible representation of G is of degree* $[G : A]$.

Theorem A(i) and the first part of Theorem A(ii) are special cases of the following:

THEOREM B. *Let $A \geqslant Z(G)$ be an abelian normal subgroup of a finite p-group G. Suppose $[G, A]$ is cyclic and $\Omega([G, A]) \leqslant Z(G)$. Then $C_A(C_G(D)) = D$ and $[G : C_G(D)] = [D : Z(G)]$ for every $Z(G) \leqslant D \leqslant A$.*

We note that in case $p > 2$, the condition $\Omega([G, A]) \leqslant Z(G)$ is automatically satisfied.

In §§3 and 4 we will prove these results by applying the double centralizer property in the theory of an Azumaya algebra B over a commutative ring (with identity) [**2**, Chapter II]: $C_B(C_B(E)) = E$ for every separable subalgebra E of B. In [**1**], a purely group-theoretical method is given to prove that if G is a finite p-group with cyclic commutator subgroup G' such that $\Omega(G') \leqslant Z(G)$, then $C_G(C_G(D)) = D$ for every $Z(G) \leqslant D \leqslant G$. In §5, we will also briefly indicate how to prove this result by the theory of Azumaya algebras.

Finally, we remark that Theorem A(ii) generalizes [**5**, III, 13.7(c) and **5**, V, 16.14]. G. A. How [**4**] has an independent proof of Theorem A(ii).

Received by the editors February 24, 1982.
1980 *Mathematics Subject Classification.* Primary 20D15.
Key words and phrases. Commutator subgroup, Azumaya algebras.

0002-9939/82/0000-0285/$02.00

2. Preliminaries. In this section, we will prove two easy lemmas (possibly known).

LEMMA 1. *Let $U \neq 1$ be a cyclic subgroup of a finite abelian group V. Then V can be embedded into the units group of a commutative ring R such that*

(i) *R contains the rational number field Q,*

(ii) *$g - 1$ is not a zero divisor for any nonidentity element $g \in U \leqslant R$.*

PROOF. First of all, V can be embedded into a finite homocyclic abelian group, say A. As U is cyclic, we may assume $A = A_1 \times A_2 \times \cdots \times A_t$ so that A_i $(1 \leqslant i \leqslant t)$ is cyclic, $U \leqslant A_1$, and $|A_1| = |A_2| = \cdots = |A_t|$. Let $A_i = \langle a_i \rangle$ for $1 \leqslant i \leqslant t$. $\{a_1 a_2 \cdots a_t, a_2, \ldots, a_t\}$ is a base for A. The coordinates of nonidentity elements of $\langle a_1 \rangle$ in this base are nonidentity. Embed $A = \langle a_1 a_2 \cdots a_t \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$ into the units group of the ring $R = \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}$, the direct sum of t copies of the complex field $\mathbf{C}$. In this embedding, it is easy to see that (i) and (ii) are satisfied.

LEMMA 2. *Let $G = \langle x, y \rangle$ be a finite p-group with cyclic commutator subgroup G' such that $\Omega(G') \leqslant C_G(x)$. Suppose $[x^p, y] = 1$. Then $[x, y]^p = 1$.*

PROOF. For convenience, let $z = [x, y]$. If $p > 2$, then G is regular and $[x^p, y] = [x, y]^p = 1$. (In fact, one can prove this more directly.)

Now let $p = 2$. The condition $[x^2, y] = 1$ is equivalent to $z^x = z^{-1}$. Assume $z \notin C_G(x)$ and $z^{2^r} \in C_G(x)$ with minimal r. That is $r \geqslant 1$ and $z^{2^{r-1}} \notin C_G(x)$. This implies $z^{2^r} \neq 1$. Now $(z^{2^r})^x = z^{-2^r} = z^{2^r}$. So $z^{2^{r+1}} = 1$. $z^{2^{r-1}}$ is of order 4 and is not in $C_G(x)$, a contradiction. Therefore, $z \in C_G(x)$ and $z^2 = 1$.

3. Proof of Theorem B. By Lemma 1, we may embed $Z = Z(G)$ into the units group of a commutative ring R such that (1) R contains the rational number field Q, and (2) $g - 1$ is not a zero divisor for any nonidentity element $g \in [G, A] \cap Z$. Let $\{x_1 = 1, x_2, \ldots, x_r\}$ be a set of coset representatives of Z in A. Denote S as a free R-module with free basis x_i, $1 \leqslant i \leqslant r$. Define a multiplication on S, distributively, as $x_i \cdot x_j = a x_k$, where x_k is the coset representative of $x_i x_j$ and $a \in Z \subseteq R$. Clearly, S is an R-algebra and A can be viewed, in the natural way, as a subgroup of units group of S. It is easy to see that the relation $ag = a$ for $a \in S$, $g \in [G, A] \cap Z$ implies $a = 0$.

$\overline{G} = G/C_G(A)$, with elements denoted as $\bar{g}$ for $g \in G$, acts on A by conjugation. For convenience, we denote the action as $\bar{g}(x) = gxg^{-1}$ for $g \in G$ and $x \in A$. We may extend the action of $\overline{G}$ to S. Then $\overline{G}$ is a subgroup of R-automorphisms of S. Now we may construct a new R-algebra B as in the classical way: First let $\{u_{\bar{g}} \mid \bar{g} \in \overline{G}\}$ be a free basis for a (left) S-module. Define multiplication in this module by letting $(au_{\bar{g}})(bu_{\bar{h}}) = a\bar{g}(b)u_{\overline{gh}}$ for all $a, b \in S$, $g, h \in G$ and extending by linearity. Of course, $\{x_i u_{\bar{g}} \mid \bar{g} \in \overline{G}, 1 \leqslant i \leqslant r\}$ is a free basis for the R-algebra B. Now, we will show that B is a central separable R-algebra.

First it is easy to show that the element

$$\frac{1}{r|\overline{G}|} \sum_{\substack{1 \leqslant i \leqslant r \\ \bar{g} \in \overline{G}}} x_i u_{\bar{g}} \otimes u_{\bar{g}^{-1}} x_i^{-1} \in B \underset{R}{\otimes} B^{\mathrm{op}}$$

is a separability idempotent for B [**2**, Chapter II]. So B is a separable R-algebra. For every $D \leqslant A$ which contains Z, let RD denote the R-subalgebra of S generated by D. As above, it is easy to show that RD is a separable R-subalgebra of B. We claim that $C_B(RD) \leqslant S\overline{C_G(D)}$, the S-subalgebra of B generated by $u_{\bar{g}}$ for $\bar{g} \in \overline{C_G(D)}$. Suppose $z = \sum_{\bar{g}\in\bar{G}} a_{\bar{g}} u_{\bar{g}} \in C_B(RD)$. If $h \notin C_G(D)$, then there is $d \in D$ with minimal order such that $[d, h^{-1}] \neq 1$. So $[d^p, h^{-1}] = 1$. By Lemma 2, $[d, h^{-1}]$ is of order p and hence is in $[G, A] \cap Z$.

$$z = d^{-1}zd = \sum a_{\bar{g}} d^{-1} u_{\bar{g}} d = \sum a_{\bar{g}} d^{-1} g d g^{-1} u_{\bar{g}} = \sum a_{\bar{g}} \left[d, g^{-1}\right] u_{\bar{g}}.$$

So $a_{\bar{h}}[d, h^{-1}] = a_{\bar{h}}$ and $a_{\bar{h}} = 0$. This proves that $C_B(RD) \leqslant S\overline{C_G(D)}$. In particular, we have $C_B(S) = C_B(RA) \leqslant S\overline{C_G(A)} = S$.

Suppose $z \in Z(B) = C_B(B)$. Then $z \in S$. Let $z = a_1x_1 + a_2x_2 + \cdots + a_rx_r$, where $a_i \in R$. For fixed i, $2 \leqslant i \leqslant r$, let g be an element of G with minimal order so that $[g^{-1}, x_i^{-1}] \neq 1$. Then $[g^{-p}, x_i^{-1}] = 1$. By Lemma 2, $[g^{-1}, x_i^{-1}]$ is of order p and hence is in $[G, A] \cap Z$.

$$zu_{\bar{g}} = u_{\bar{g}}z = \sum_{j=1}^{r} a_j u_{\bar{g}} x_j = \sum_{j=1}^{r} a_j g x_j g^{-1} u_{\bar{g}} = \sum_{j=1}^{r} a_j \left[g^{-1}, x_j^{-1}\right] x_j u_{\bar{g}}.$$

So $a_i[g^{-1}, x_i^{-1}] = a_i$ and $a_i = 0$. Therefore, $z \in R$ and $Z(B) = R$. That is, B is a central separable R-algebra.

We note that S is actually a Galois extension of R with Galois group $\bar{G}$ and $B = \Delta(S : \bar{G})$ in the notation of [**2**, Chapter III]. Since we do not need this fact, we will not prove it here.

Now, for every $Z \leqslant D \leqslant A$, by double centralizer properties in the theory of Azumaya algebras [**2**, Chapter II], we have $RD = C_B(C_B(RD))$. Then

$$RD = C_B(C_B(RD)) \geqslant C_B\left(S\overline{C_G(D)}\right) \geqslant R\left(C_G(C_G(D)) \cap A\right).$$

As $D = RD \cap A \geqslant C_G(C_G(D)) \cap A = C_A(C_G(D)) \geqslant D$, so $D = C_A(C_G(D))$. This proves the first result in Theorem B.

To prove the second result, let $|A/Z| = p^n$ and $|D/Z| = p^r$ and take a series $Z = D_0 < D_1 < \cdots < D_r = D < D_{r+1} < \cdots < D_n = A$, with $[D_{i+1} : D_i] = p$ $(0 \leqslant i < n)$. Then

$$G = C_G(D_0) \geqslant C_G(D_1) \geqslant \cdots \geqslant C_G(D_n) = C_G(A).$$

Applying $C_A(\cdot)$ to the above series, we get the original series. So $[C_G(D_i) : C_G(D_{i+1})] = p$ for all i. Hence $[G : C_G(D)] = p^r$ and $[G : C_G(D)] = [D : Z]$. This completes the proof of the theorem.

4. Proof of Theorem A and a corollary. Theorem A(i) and the first part of Theorem A(ii) follow easily from Theorem B. Now let σ be a faithful absolutely irreducible representation of G over a field F. σ maps G into $M = \mathrm{Mat}_n(F)$, the full matrix ring of degree n over F. Let $\{x_1 = 1, x_2, \ldots, x_r\}$ be a set of coset representatives of Z in A. Let $\{y_1 = 1, y_2, \ldots, y_r\}$ be a set of right coset representatives of A in G. We first claim that $\{\sigma(x_iy_j) \mid 1 \leqslant i, j \leqslant r\}$ is a linearly independent set over F and $F\sigma(G)$ is a central separable F-algebra. In fact, the proof is the same as the method we used to prove that B is central in Theorem B. Here we omit the detail. Since σ is absolutely irreducible over F, $C_M(F\sigma(G)) = F$. Then

$$n^2 = [M : F] = [F\sigma(G) : F][C_M(F\sigma(G)) : F] = [F\sigma(G) : F] = [G : Z].$$

This completes the proof of Theorem A.

As a corollary to Theorem A(ii) and [**6**, Theorem 8], we get

COROLLARY. *Let A be a maximal abelian normal subgroup of a finite p-group G. Suppose $[G, A]$ is cyclic and $\Omega([G, A]) \leqslant Z(G)$. Then all maximal abelian normal subgroups of G are of order $|Z(G)|[G : Z(G)]^{1/2}$. If, in addition, $p > 2$, then all maximal abelian subgroups of G are of order $|Z(G)|[G : Z(G)]^{1/2}$.*

5. Finite p-groups with cyclic commutator subgroup. In this section, we will briefly indicate how to apply the method we used in §3 to the finite p-groups with cyclic commutator subgroup.

THEOREM C [**1**, THEOREM 2]. *Let G be a finite p-group with cyclic commutator subgroup G'. Suppose $\Omega(G') \leqslant Z(G)$. Then $C_G(C_G(D)) = D$ for every $Z(G) \leqslant D \leqslant G$.*

PROOF. As in §2, we embed $Z = Z(G)$ into a good commutative ring R so that $g - 1$ is not a zero divisor for every $g \in G' \cap Z$. Let $\{g_1 = 1, g_2, \ldots, g_s\}$ be a set of coset representatives of Z in G. Let this set be a free R-basis for an R-module B. Define a multiplication on B so that B forms an R-algebra and G can be viewed as in the units group of B. For every $Z \leqslant D \leqslant G$, let RD be the R-subalgebra of B generated by D. By the same method we used in §3, we can obtain that RD is a separable R-algebra and $C_B(RD) = RC_G(D)$. In particular, $C_B(B) = C_B(RG) = RC_G(G) = R$. So B is an Azumaya algebra over R. As RD is a separable R-algebra, we get

$$RD = C_B(C_B(RD)) = C_B(RC_G(D)) = RC_G(C_G(D)).$$

By taking the intersection with G, we get $D = C_G(C_G(D))$. This completes the proof.

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