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THE ENDOMORPHISM RING OF AN ARTINIAN MODULE WHOSE HOMOGENEOUS LENGTH IS FINITE

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ABSTRACT. Smalø [2] showed that the index of nilpotency of the endomorphism ring of a module M_R of finite length is bounded by the number $\max\{n_A \mid A_R \text{ simple}\}$, where n_A denotes the number of times A_R occurs as a factor in a composition chain of M_R. We give another proof of Smalø's theorem which leads to an analogous result for artinian modules whose homogeneous length is finite.

Let M_R be a (semi-)artinian module. Then M_R possesses an ascending composition chain, i.e. a chain $\{M_i\}_{i\leqslant a}$ of submodules of M_R indexed by ordinals, having the properties $M_0 = 0$, $M_a = M$, M_{i+1}/M_i simple for all $i < a$ and $M_j = \bigcup_{i<j} M_i$ for all limit ordinals $j \leqslant a$. The following infinite version of the Jordan-Hölder Theorem shows that any two ascending composition chains of M_R are isomorphic.

PROPOSITION 1. *Given two ascending composition chains $\{M_i\}_{i\leqslant a}$ and $\{N_j\}_{j\leqslant b}$ of M_R, there is a bijection v: $a \to b$ such that $M_{i+1}/M_i \cong N_{v(i)+1}/N_{v(i)}$ as R-modules for all $i < a$.*

PROOF. Let $S(a)$ and $S(b)$ be the sets of successors of the elements in a and b. Define a mapping $\tilde{v}$: $S(a) \to S(b)$ by $\tilde{v}(i+1) = \min\{k \mid M_{i+1} \subset N_k + M_i\}$ for all $i < a$. Then $\tilde{v}$ is well defined, as the minimum exists and is a successor ordinal $\leqslant b$. Conversely, define $\tilde{w}$: $S(b) \to S(a)$ by $\tilde{w}(j+1) = \min\{h \mid N_{j+1} \subset M_h + N_j\}$. Then $\tilde{v}$ and $\tilde{w}$ are inverse to each other, and M_{i+1}/M_i is isomorphic to $N_{\tilde{v}(i+1)}/N_{\tilde{v}(i+1)-1}$ for all $i < a$. Hence, the mapping v: $a \to b$, defined by $v(i) = \tilde{v}(i+1) - 1$, has the desired properties.

Let A_R be a simple module. The A-length of M_R is defined to be the cardinality of the set $\{i < a \mid M_{i+1}/M_i \cong A_R\}$, and the homogeneous length of M_R, denoted by $\mathfrak{hl}(M_R)$, is defined to be the supremum of the A-lengths of M_R, where A runs through a complete system of simple R-modules. By Proposition 1, the A-length and the homogeneous length of M_R are invariants of the (semi-)artinian module M_R. We note that, if $U \subset M_R$, the A-length of M is the sum of the A-lengths of U and M/U. Assume now M_R to be artinian with $\mathfrak{hl}(M_R)$ finite. Our aim is to prove that $S = \operatorname{End}(M_R)$ is semiprimary, and that the index of nilpotency of $Ra(S)$ is bounded by $\mathfrak{hl}(M_R)$.

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PROPOSITION 2. *Let M_R be artinian with $\mathfrak{hl}(M_R)$ finite. Then $S = \mathrm{End}(M_R)$ is semiprimary.*

PROOF. Without loss of generality, let M_R be indecomposable. Using a slight variation of the proof of Fitting's lemma, we first show that any noninvertible $f \in S$ is nilpotent: As M_R is artinian, the equalities $M = \mathrm{Im}\, f^n + \mathrm{Ker}\, f^n$ and $\mathrm{Im}\, f^{2n} = \mathrm{Im}\, f^n$ hold for some $n \in \mathbf{N}$. Assume there exists a nonzero $m \in \mathrm{Im}\, f^n \cap \mathrm{Ker}\, f^n$. Let A_R be any simple composition factor of mR. Then the A-length of $f^n\,(\mathrm{Im}\, f^n)$ is less than the A-length of $\mathrm{Im}\, f^n$, a contradiction. Consequently, $M = \mathrm{Im}\, f^n \oplus \mathrm{Ker}\, f^n$, and $f^n = 0$.

As all noninvertible elements of S are nilpotent, S is a local ring, and $Ra(S)$ is nil. Hence, $Ra(S)$ is nilpotent by Fisher [**1**, Theorem 1.5].

LEMMA 3. *Let ${}_SX_R$ be a bimodule. Let N be a subset of S with the property that for every $x \in X$ there is a finite subset N_x of N such that $r(N) \cap xR = r(N_x) \cap xR$. ($r(N)$ denotes the right annihilator of N in X.) Then, if X_R has a composition factor isomorphic to some simple module A_R, NX or $r(N)$ does.*

PROOF. Let $x \in X$ with $xR/K \cong A_R$. If $r(N) \cap xR \not\subset K$, then we are done. If not, let $r(N) \cap xR = r(n_1,\ldots,n_k) \cap xR$. Then the map $g\colon xR \to \prod_{i=1}^k n_i xR$, $g(xr) = (n_1xr,\ldots,n_kxr)$, has kernel $r(N) \cap xR \subset K$. Therefore, $\mathrm{Im}\, g$ has a composition factor isomorphic to A_R, hence one of the n_ixR and NX do.

LEMMA 4. *Let ${}_SX_R$ be a bimodule with S semiprimary and X_R artinian. Then, if X_R has a composition factor isomorphic to some simple module A_R, $So({}_SX)_R$ does.*

PROOF. $N = Ra(S)$. As X_R is artinian, for every $x \in X$ there is a finite subset N_x of N such that $r(N) \cap xR = r(N_x) \cap xR$. By Lemma 3, NX or $r(N)$ has a composition factor isomorphic to A_R. If NX does, we are done by induction over the Loewy length of ${}_SX$, otherwise $r(N) = So({}_SX)$ does.

THEOREM 5. *Let M_R be artinian with $\mathfrak{hl}(M_R)$ finite. Then $S = \mathrm{End}(M_R)$ is semiprimary, and the index of nilpotency of $N = Ra(S)$ is less than or equal to $\mathfrak{hl}(M_R)$.*

PROOF. The first assertion was proved in Proposition 2. Consider now the ascending Loewy chain $0 \underset{\neq}{\subset} r(N) \cdots r(N^{h-1}) \underset{\neq}{\subset} r(N^h) = M$ of ${}_SM$. The index of nilpotency of N equals h, because ${}_SM$ is faithful. Let A_R be a simple composition factor of $M/r(N^{h-1})$. If $X^i = M/r(N^i)$ $(i = 0,\ldots,h-1)$, then, by Lemma 4, the module $So({}_SX^i)_R = r(N^{i+1})/r(N^i)$ contains a simple composition factor isomorphic to A_R for $i = 0,\ldots,h-1$, and the second assertion is proved.

As a corollary of Theorem 5, we obtain Smalø's theorem as cited in the abstract.

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