PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

RULED FUNCTION FIELDS

JAMES K. DEVENEY

ABSTRACT. Let $L = L_1(x_1) = L_2(x_2) \supset K$ where x_i is transcendental over L_i, and L_i is a finitely generated transcendence degree 1 extension of K, $i = 1, 2$. If the genus of $L_1/K = 0$, then L_1 and L_2 are K-isomorphic. If the genus of $L_1/K > 0$, then $L_1 = L_2$ and moreover L_1 is invariant under all automorphisms of L/K. A criterion is also established for a subfield of a ruled field L to be ruled.

Let L be a finitely generated extension of a field K. L is said to be ruled over K if there exists an intermediate field L_1 and an element x_1 transcendental over L_1 such that $L = L_1(x_1)$. The Zariski problem [6] asks: If $L = L_1(x_1) = L_2(x_2)$ is ruled in two ways over K, must L_1 and L_2 be K-isomorphic? The answers to some special cases of the 1-dimensional problem were announced in [6] and here we provide a complete affirmative answer for the 1-dimensional case. Henceforth, we assume the transcendence degree of L over K is 2. If L_1 is an intermediate field of L/K of transcendence degree 1 over K, the genus of L_1/K is by definition the genus of L_1 over the algebraic closure of K in L_1.

The proof of the one dimensional case is achieved by examining the possibilities for L to be ruled over two distinct subfields L_1 and L_2. If $L_1 \cap L_2 = K$, then L_1 and L_2 must be of genus 0. This leads to the result that if $L = L_1(x_1) \supset L_1 \supset K$ with x_1 transcendental over L_1 and the genus of L_1/K is positive, then L_1 must be invariant under all automorphisms of L/K. This result is then used to establish sufficient conditions for a subfield of a ruled field to be ruled (K not necessarily algebraically closed). Recall, L is regular over K means L is separable over K, and K is algebraically closed in L.

PROPOSITION 1. *Suppose $L = L_1(x_1) = L_2(x_2) \supset K$ where x_i is transcendental over L_i, and L_i is a finitely generated transcendence degree 1 extension of K, $i = 1, 2$. If $L_1 \cap L_2 = K$, then L_1 and L_2 are K-isomorphic genus 0 extensions of K.*

PROOF. Since each L_i is algebraically closed in L, the algebraic closure of K in L is contained in each L_i. Thus K is algebraically closed in L since $L_1 \cap L_2 = K$. By [4, Theorem 1.1, p. 1304], there exists a unique minimal intermediate field L^* over which L is separable. Since L is separable over L_1 and L_2, $L^* \subseteq L_1 \cap L_2$. Thus $L^* = K$, i.e., L is separable over K. Thus each L_i is separable, hence regular, over K. Since $L_1 \cap L_2 = K$, we have $L_1 \not\subseteq L_2$; and therefore some element of L_1 is transcendental over L_2. Since the transcendence degree of L_1/K is 1, a transcendence basis for L_1/K remains independent over L_2, i.e., L_1 and L_2 are free over K. By [5, Theorem 3, p. 57], L_1 and L_2 are linearly disjoint over K. Now, $L_2(x_2) \supseteq L_2L_1 \supset L_2$, and hence by Luroth's theorem, L_2L_1 is simple transcendental

Received by the editors October 15, 1981 and, in revised form, January 29, 1982.
1980 *Mathematics Subject Classification*. Primary 14J25; Secondary 12F20.
Key words and phrases. Ruled field, genus.

over L_2. Thus L_2L_1 is of genus 0 over L_2. By [**3**, Theorem 2, p. 132], L_1/K is of genus 0. By a symmetric argument, L_2/K is also of genus 0.

Recall that a genus 0 extension L_1 of a finite field K is pure transcendental. One sees this as follows: since L_1 has a divisor of degree 1 [**3**, Theorem, p. 148] and since the genus is 0, the corollaries to the Riemann-Roch theorem [**3**, p. 40] show this divisor must be integral, hence a prime divisor of degree 1, and hence L_1/K is simple transcendental [**3**, Theorem, p. 50]. Thus if K is finite, L_1 and L_2 are simple transcendental extensions of K, and hence are isomorphic.

If K is infinite, [**7**, Lemma 1, p. 209] shows L_1 is K-isomorphic to a subfield of L_2, and L_2 is K-isomorphic to a subfield of L_1. If L_1 is simple transcendental over K, then so is L_2 by Luroth's theorem. If L_1 is not simple transcendental over K, then [**1**, Corollary 11.3, p. 42] shows L_1 and L_2 are K-isomorphic. Q.E.D.

It should be noted that if L_1 is a nonrational genus 0 function field over $K(\operatorname{char} K \neq 2)$ and L_2 is a K-isomorphic copy of L_1, then L_2L_1, the free join of L_1 and L_2, will be ruled over both L_1 and L_2 [**1**, p. 41].

THEOREM 2. *Suppose $L = L_1(x_1) = L_2(x_2) \supset K$ where x_i is transcendental over L_i, and L_i is a finitely generated transcendence degree 1 extension of K, $i = 1,2$. Then L_1 and L_2 are K-isomorphic.*

PROOF. It suffices to show they are isomorphic over their intersection, which contains K. If their intersection is algebraic over K, then Proposition 1 applies. If it is not algebraic over K, then each of L_1 and L_2 must be the algebraic closure in L of their intersection. Thus they are equal in this case.

THEOREM 3. *Suppose $L = L_1(x_1) \supset L_1 \supset K$ where x_1 is transcendental over L_1 and L_1 is a finitely generated transcendence degree 1 extension of K. Assume the genus of $L_1/K > 0$. Then L_1 is invariant under any K-automorphism of L.*

PROOF. Let α be a K-automorphism of L. Then $L = L_1(x_1) = L_1^\alpha(x_1^\alpha)$. Since L_1/K is not of genus 0, Proposition 1 shows $L_1 \cap L_1^\alpha$ cannot be algebraic over K. But then L_1 and L_1^α are both the algebraic closure of $L_1 \cap L_1^\alpha$ in L, i.e., $L_1 = L_1^\alpha$. Q.E.D.

If L is ruled over K, must an intermediate field F with $[L:F] < \infty$ also be ruled over K? If K is algebraically closed of char 0, [**2**, Proposition 2, p. 106] shows the answer is yes. For K not algebraically closed (but still of char 0), the answer is no. An example is given in [**8**, p. 330]. There, $K = C(\mu)$, $L = C(\mu, v, w)$ where $\{\mu, v, w\}$ is algebraically independent over C. A subfield F with $[C(\mu,v,w):F] = 2$ is constructed with F not ruled over $C(\mu)$. Actually, [**2**] shows F is not pure transcendental over $C(\mu)$. However, if F were ruled, then F would be pure transcendental by the generalized Luroth theorem [**6**]. However, we can use the results of this paper to get an affirmative answer in some cases.

THEOREM 4. *Let $L = L_1(x_1) \supset L_1 \supset K$ where L_1 is a finitely generated extension of K of transcendence degree 1 and positive genus with x_1 transcendental over L_1. Let G be a finite group of K-automorphisms of L and let F be its fixed field. If $|G|$ is odd, then F is also ruled over K.*

PROOF. Since L_1 is invariant under the action of G by Theorem 3, it follows from [**8**, Theorem 4, p. 322] that F is pure transcendental over $F \cap L_1$.

References

1. S. Amitsur, *Generic splitting fields of central simple algebras*, Ann. of Math. (2) **62** (1955), 8–43.
2. S. Arima, *Double ruled surfaces and their canonical systems*, J. Math. Soc. Japan **22** (1970), 105–112.
3. M. Deuring, *Lectures on the theory of algebraic functions of one variable*, Lecture Notes in Math., vol. 314, Springer-Verlag, Berlin and New York, 1973.
4. J. Deveney and J. Mordeson, *Subfields and invariants of inseparable field extensions*, Canad. J. Math. **29** (1977), 1304–1311.
5. S. Lang, *Introduction to algebraic geometry*, Interscience Tracts in Pure and Appl. Math., vol. 5, Interscience, New York, 1958.
6. M. Nagata, *A theorem on valuation rings and its applications*, Nagoya Math. J. **29** (1967), 85–91.
7. P. Roquette, *Isomorphisms of generic splitting fields of simple algebras*, J. Reine Angew. Math. **214–215** (1964), 207–226.
8. D. Triantaphyllou, *Invariants of finite groups acting non-linearly on rational function fields*, J. Pure Appl. Algebra **18** (1980), 315–331.

Department of Mathematical Sciences, Virginia Commonwealth University, Richmond, Virginia 23284