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EXTREME POINTS AND $l_1(\Gamma)$-SPACES

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ABSTRACT. Let X be a nontrivial real Banach space and let E_X denote the set of extreme points of the closed unit ball $B(X)$.

THEOREM 1. *X is an $l_1(\Gamma)$-space if and only if* (i) span(e) *is an L-summand in X for every e in E_X and* (ii) $B(X)$ *is the norm closed convex hull of E_X.*

THEOREM 2. *Let $X = Y^*$. If* (i) span(e) *is an L-summand in X for every e in E_X and* (ii) $\{e \in E_X: e(y) = 1\}$ *is countable for each y in Y with $\|y\| = 1$, then X is an $l_1(\Gamma)$-space.*

By definition, an L-projection on a Banach space X is a projection P such that $\|x\| = \|Px\| + \|x - Px\|$ for every x in X; the range of P is called an L-summand in X. An $l_1(\Gamma)$-space is a Banach space which is linearly isometric to the space $l_1(\Gamma)$ of all real-valued summable functions on some set Γ. Let X be a nontrivial real Banach space and let E_X denote the set of extreme points of the closed unit ball $B(X)$. In this paper we prove (Theorem 1) that X is an $l_1(\Gamma)$-space if and only if (i) span(e) is an L-summand in X for every e in E_X and (ii) $B(X)$ is the norm closed convex hull of E_X. As a consequence we have (Theorem 2) that a dual space $X = Y^*$ is an $l_1(\Gamma)$-space if (i) span(e) is an L-summand in X for every e in E_X and (ii) $\{e \in E_X: e(y) = 1\}$ is countable for each y in Y with $\|y\| = 1$. The proof of Theorem 2 uses the Bishop-Phelps theorem and a result of J. Bourgain to show that $B(X)$ is the norm closed convex hull of E_X. Our paper concludes with an example of a nonseparable space Y which satisfies the hypotheses of Theorem 2 and contains uncountably many y such that $\|y\| = 1$ and $\{e \in E_{Y^*}: e(y) = 1\}$ is countably infinite.

In what follows, if S is a subset of a Banach space, then the convex hull of S is denoted by $\operatorname{co} S$ and the linear span of S by $\operatorname{span} S$. The norm closure of S is denoted by norm-cl(S). All Banach spaces are assumed to be nontrivial.

In Lemmas 1 and 2, X is a real Banach space for which $E_X \neq \varnothing$.

LEMMA 1. *Let A be a nonempty finite subset of E_X such that* span(e) *is an L-summand in X for every e in A, and let $N =$* span A. *Then* $B(N) = \operatorname{co}(A \cup -A)$.

PROOF. Since $N = \sum \operatorname{span}(e)$ ($e \in A$), we have that N is an L-summand in X and $E_N = A \cup -A$ [**1**, Propositions 1.13 and 1.15]. Then $B(N) = \operatorname{co}(A \cup -A)$ because A is finite.

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LEMMA 2. *Assume that* span(e) *is an L-summand in X for every e in* E_X. *Let* $\{e_n: n = 1,2,\ldots\}$ *be a linearly independent subset of* E_X *and let* $x_n \in \text{span}(e_n)$ *for* $n = 1,2,\ldots$. *If* $\Sigma\|x_n\| < \infty$, *then* Σx_n *converges and* $\|\Sigma x_n\| = \Sigma\|x_n\|$.

PROOF. The proof follows from the fact that $\|\Sigma_{n=1}^k x_n\| = \Sigma_{n=1}^k \|x_n\|$ for all k. To obtain the induction step, observe that if P is the L-projection of X onto $N_k = \Sigma_{n=1}^k \text{span}(e_n)$, then $Pe_{k+1} = 0$ because $e_{k+1} \notin N_k$ by Lemma 1. (An L-projection maps an extreme point to itself or 0.)

THEOREM 1. *A real Banach space X is an* $l_1(\Gamma)$-*space if and only if* (i) span(e) *is an L-summand in X for every e in* E_X *and* (ii) $B(X)$ *is the norm closed convex hull of* E_X.

PROOF. Suppose that X is an $l_1(\Gamma)$-space. We may assume that $X = l_1(\Gamma)$, where Γ is a nonempty set. For each γ in Γ let δ_γ be the characteristic function of $\{\gamma\}$. Then $E_X = \{\pm\delta_\gamma: \gamma \in \Gamma\}$. For each γ in Γ, the map $x \mapsto x\delta_\gamma$ is an L-projection of X onto span(δ_γ). Thus condition (i) holds (as it does in any L_1-space). To prove (ii), let $x \in X$ with $\|x\| \leq 1$. Then there is a countable set $\{\gamma_n\} \subseteq \Gamma$ such that $x(\gamma) = 0$ for $\gamma \notin \{\gamma_n\}$ and $\Sigma_{n=1}^\infty |x(\gamma_n)| \leq 1$. Then $x = \Sigma_{n=1}^\infty x(\gamma_n)\delta_{\gamma_n}$. For each k let $x_k = \Sigma_{n=1}^k x(\gamma_n)\delta_{\gamma_n}$. Then $\|x_k\| \leq 1$ and hence by Lemma 1, $x_k \in \text{co}\, E_X$. Therefore $x \in \text{norm-cl}(\text{co}\, E_X)$.

For the converse, assume that (i) and (ii) are true. Let Γ be a maximal linearly independent subset of E_X. Then $E_X = \Gamma \cup -\Gamma$. To see this, suppose there is $e \in E_X$ with $e \notin \Gamma \cup -\Gamma$. Then e is a linear combination of the elements of a finite subset A of Γ. By Lemma 1, $e \in \text{co}(A \cup -A)$. Then $e \in A \cup -A$ since $e \in E_X$, and we have a contradiciton. If $\Gamma = \{e_\gamma\}$, define an operator $T: l_1(\Gamma) \to X$ by $T(f) = \Sigma f(\gamma)e_\gamma$. By Lemma 2, T is an isometry. Hence its range is closed. By (ii) and the fact that $E_X = \Gamma \cup -\Gamma$, the range of T is dense in X. Thus T is surjective.

THEOREM 2. *Let Y be a real Banach space such that* (i) span(e) *is an L-summand in* Y^* *for every e in* E_{Y^*}, *and* (ii) $\{e \in E_{Y^*}: e(y) = 1\}$ *is countable for each y in Y with* $\|y\| = 1$.

Then Y^* *is an* $l_1(\Gamma)$-*space*.

PROOF. By Theorem 1 it suffices to show that $B(Y^*)$ is the norm closed convex hull of E_{Y^*}. Let $f \in B(Y^*)$ with $f \neq 0$. By the Bishop-Phelps theorem [2], the set of those g in Y^* which attain their norm is dense in Y^*. Hence given $\varepsilon > 0$, there is g in Y^* such that $\|f/\|f\| - g/\|g\|\| < \varepsilon$ and $\|g\| = g(y)$, where $y \in Y$ with $\|y\| = 1$. Let $F_y = \{h \in B(Y^*): h(y) = 1\}$. Then F_y is a weak* compact convex set and $g/\|g\| \in F_y$. Let E_y denote the set of extreme points of F_y. Then $E_y \subseteq E_{Y^*}$ because F_y is an extremal subset of $B(Y^*)$. Thus $E_y = \{e \in E_{Y^*}: e(y) = 1\}$. Then $F_y = \text{norm-cl}(\text{co}\, E_y)$ because E_y is countable [3]. Let $h \in \text{co}\, E_y$ with $\|h - g/\|g\|\| < \varepsilon$. Then $\|h - f/\|f\|\| < 2\varepsilon$, hence

$$\|\,\|f\|h - f\| < 2\varepsilon\|f\|.$$

Since $\|f\|h \in \text{co}(E_y \cup -E_y)$, it follows that $f \in \text{norm-cl}(\text{co}\, E_{Y^*})$.

We now give an example of a space Y which satisfies the hypotheses of Theorem 2 and contains uncountably many y such that $\|y\| = 1$ and $\{e \in E_{Y^*}: e(y) = 1\}$ is countably infinite.

Let T denote the set of all ordinals less than or equal to the first uncountable ordinal Ω, and let T have the order topology. Let $Y = \{f \in C(T): f(\Omega) = 0\}$. Then Y^* is an L-space because Y is an M-space; hence the first hypothesis of Theorem 2 is satisfied. For each t in T, let the evaluation functional e_t be defined on Y by $e_t(f) = f(t)$ for all f in Y. Then $E_{Y^*} = \{\pm e_t: t \in T, t \neq \Omega\}$. Since each function in $C(T)$ is eventually constant, the second hypothesis of Theorem 2 is satisfied. For each t in T such that $\omega \leqslant t < \Omega$, let f_t be the characteristic function of the interval $[0, t]$. Then $f_t \in Y$, $\|f_t\| = 1$, and $\{e \in E_{Y^*}: e(f_t) = 1\}$ is countably infinite. Clearly the set of functions f_t is uncountable.

In conclusion, we remark that $C(T)^* = l_1(T)$ [**4**, p. 175], hence the converse of Theorem 2 is false.

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