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ELEMENTARY PROOFS OF SOME ASYMPTOTIC RADIAL UNIQUENESS THEOREMS

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ABSTRACT. Elementary proofs of several generalizations of Tse's extension of an asymptotic radial uniqueness theorem of Barth and Schneider are given.

Let $\Delta = \{|z|< 1\}$ and $C = \{|z|= 1\}$. The following is an extension to meromorphic functions by Tse [**5**] of a theorem of Barth and Schneider [**1**].

THEOREM 1. *Let μ be a positive monotone decreasing function with domain* $[0, 1)$ *such that* $\lim_{r\to 1}\mu(r) = 0$. *Let S be a second category subset of C. If f is a meromorphic function on Δ with the property that $f(r\eta) = o[\mu(r)]$ for each $\eta \in S$, then $f \equiv 0$.*

Barth and Schneider's proof (for bounded analytic functions) depends on deep theorems of Mergelyan, Lusin-Privalov, and Collingwood. Tse's proof of Theorem 1 is based on the method of Barth and Schneider. An equivalent formulation [**4**, Corollary 2] of Theorem 1 may be obtained in a more straightforward manner as a corollary of a theorem of Rippon [**4**, Theorem 1]; however, this proof still relies on the Collingwood maximality theorem as well as results and methods needed for a proof of the Lusin-Privalov theorem. The proofs that we shall give to several generalizations of Theorem 1 are elementary and are based on the following.

CATEGORY PRINCIPLE. *Let S be a second category set. If $S = \bigcup_{n=1}^{\infty} F_n$ with each F_n closed, then some F_n contains a nonempty open set.*

PROOF. Since E is of second category, some F_n must be dense in an open set U. Since F_n is closed, we have $U \subseteq F_n$ as required.

We turn now to our first generalization of Theorem 1. Let $\mathfrak{G}$ be a continuum contained in $\bar{\Delta} = \{|z| \leq 1\}$ such that $\mathfrak{G} \cap C = \{1\}$ and let $\mathfrak{G}_\eta = \{\eta z: z \in \mathfrak{G}\}$ for each $\eta \in C$. For μ a positive function with domain $[0, 1)$ such that $\lim_{r\to 1}\mu(r) = 0$ and $\eta \in C$, we shall write $f(z) = O[\mu(|z|)]$, $z \in \mathfrak{G}_\eta$, when $\limsup_{|z|\to 1} |f(z)|/\mu(|z|) < +\infty$, $z \in \mathfrak{G}_\eta \cap \Delta$.

THEOREM 2. *Let μ be a positive function with domain* $[0, 1)$ *such that* $\lim_{r\to 1}\mu(r) = 0$ *and S a second category subset of C. If f is a meromorphic function on Δ with the property that $f(z) = O[\mu(|z|)]$, $z \in \mathfrak{G}_\eta$, for each $\eta \in S$, then $f \equiv 0$.*

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Theorem 1 follows from Theorem 2 when $\mathfrak{G} = [0, 1]$.

PROOF. Let $Z_\mu(f) = \{\eta \in C: f(z) = O[\mu(|z|)], z \in \mathfrak{G}_\eta\}$. Then $Z_\mu(f) = \bigcup_{n=1}^\infty F_n$ where $F_n = \{\eta \in C: |f(z)| \leq n\mu(|z|), z \in \mathfrak{G}_\eta \text{ and } 1 - \frac{1}{n} \leq |z| < 1\}$ for each n. Now each F_n is closed by the continuity of f. Since $Z_\mu(f)$ is of second category by assumption, there exists some n for which F_n contains a nonempty open arc A (category principle). Note that the set $\bigcup_{\eta\in A} \mathfrak{G}_\eta \cap \{1 - \frac{1}{n} \leq |z| < 1\}$ is a neighborhood (in Δ) of each point of A. It follows from the definition of F_n that f is continuously 0 at each point of the arc A. We conclude from the Schwarz reflection principle and the identity theorem that $f \equiv 0$. Theorem 2 is established.

Our next generalization extends Theorem 1 to the unit ball in $\mathbf{C}^n$, $n \geq 1$. Let $\Delta_n = \{z \in \mathbf{C}^n: \|z\| < 1\}$ and $C_n = \{z \in \mathbf{C}^n: \|z\| = 1\}$.

THEOREM 3. *Let μ be as in Theorem 2 and S a second category subset of C_n. If f is meromorphic on Δ_n with $f(r\eta) = O[\mu(r)]$ for each $\eta \in S$, then $f \equiv 0$.*

PROOF. Except for the modification that A is now an open subset of C_n instead of an open arc of C, the proof proceeds as above (with $\mathfrak{G} = [0, 1]$) up to the conclusion that f is continuously 0 at each point of A.

We show that $f \equiv 0$ as follows. Let $w \in \Delta_n$ and $\eta \in A$. There exists a univalent analytic map φ defined on a neighborhood of $\overline{\Delta}_n$ mapping Δ_n onto itself such that $\varphi(w) = (0,\ldots,0)$ and $\varphi(\eta) = (1,0,\ldots,0)$. (Such a map φ can be constructed explicitly using maps of the form

$$\psi_\beta: (z_1,\ldots,z_n) \to \left(\frac{z_1 - \beta}{1 - \bar{\beta}z_1}, \frac{\sqrt{1-|\beta|^2}}{1-\bar{\beta}z_1} z_2, \ldots, \frac{\sqrt{1-|\beta|^2}}{1-\bar{\beta}z_1} z_n \right),$$

$(z_1,\ldots,z_n) \in \Delta_n$ for $\beta \in \Delta$ and unitary linear transformations; cf. [**2**, p. 420].) Then $g(z) = f \circ \varphi^{-1}(z,0,\ldots,0)$ is a meromorphic function on Δ which is continuously 0 at each point of an open arc I containing 1 and contained in $\{\xi \in C: (\xi,0,\ldots,0) \in \varphi(A)\}$. It follows from the Schwarz reflection principle and the identity theorem that $g \equiv 0$. In particular, $f(w) = f \circ \varphi^{-1}(0,\ldots,0) = g(0) = 0$. Since $w \in \Delta_n$ was arbitrary, we conclude that $f \equiv 0$. This completes the proof of Theorem 3.

It is possible to generalize Theorem 3 in a way analogous to that in which Theorem 2 generalizes Theorem 1, though care must be taken in framing a workable definition of rotating a continuum when $n > 1$. One possibility is to phrase such a definition in terms of group actions on the sphere C_n. An analogue of such a generalization for a half space $H_n = \{(z_1,\ldots,z_n) \in \mathbf{C}^n: \operatorname{Im} z_n > 0\}$ is more easily formulated and proved; however, we shall not pursue these generalizations here.

Rippon [**3**, Theorem 3] has given a subharmonic analogue of Theorem 1 for the half space $D = \{(x_1,\ldots,x_n) \in \mathbf{R}^n: x_n > 0\}$ when $n > 2$. His proof depends on a generalized form of the Collingwood maximality theorem for "fine continuous" functions (a class containing the subharmonic functions) proved in the same paper [**3**, Theorem 1]. It is also noted that his arguments apply equally well to the half plane. The following analogue of Rippon's result for continuous subharmonic functions in the disk Δ may be proved along the same lines as our proof of Theorem 2. We assume that $\mathfrak{G}$ and $\mathfrak{G}_\eta$, $\eta \in C$, are as preceding that theorem.

THEOREM 4. *Let ν be a real-valued function with domain* $[0, 1)$ *such that* $\lim_{r\to 1} \nu(r) = -\infty$. *Let u be a continuous subharmonic function on Δ. If S is a second category subset of C such that*

$$\limsup_{\substack{|z|\to 1 \\ z\in \mathfrak{S}_\eta \cap \Delta}} u(z) - \nu(|z|) < +\infty, \qquad \eta \in S, \tag{1}$$

then $u \equiv -\infty$.

For the proof, note that if a subharmonic function u on Δ is continuously $-\infty$ at each point of a nonempty open arc A of C, then $u \equiv -\infty$ as is seen using simple harmonic measure estimates. When f is analytic, Theorem 2 is easily subsumed under Theorem 4. In fact, letting $\nu = \log \mu$, $u = \log|f|$, and observing that $f(z) = O[\mu(|z|)]$, $z \in \mathfrak{S}_\eta$, for each $\eta \in S$ implies (1), we see that the conclusion $u \equiv -\infty$ guarantees that $f \equiv 0$. Finally, we remark that in the case when $\mathfrak{S}$ is the image of a Jordan arc γ such that $|\gamma|$ is strictly increasing, Theorems 2 and 4 are seen to be sharp when μ and ν are monotonic using only a slight modification of the construction used by P. Gauthier (see [5, Theorem B]) to show that Theorem 1 is sharp.

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