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HOLOMORPHIC MAPPINGS OF DOMAINS WITH GENERIC CORNERS

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ABSTRACT. The boundary behavior of a biholomorphic mapping f between two domains with real analytic, generic, nondegenerate corners in $\mathbf{C}^n$ is considered. Under certain minimal regularity assumptions on f it is shown that f continues holomorphically past the boundary.

Introduction. The problem of extending a holomorphic mapping between two domains with smooth boundaries in the complex space $\mathbf{C}^n$ has received considerable attention in recent years. In this note we consider the continuation problem for a mapping f defined on a domain which has corners of a certain kind. We shall show that f can be analytically extended by means of a reflection principle, provided it satisfies certain minimal initial regularity conditions. The main point here is that an argument due to H. Lewy, when suitably modified, gives holomorphic continuation in a much more general situation.

Let D be a domain in $\mathbf{C}^n$ and U an open set which meets the boundary of D. Let $r^i(z)$, $1 \leqslant i \leqslant l$, where $1 \leqslant l \leqslant n$, be twice continuously differentiable real valued functions defined on U for which $dr^1 \wedge \cdots \wedge dr^l \neq 0$ and

$$D \cap U = \{z \in U\colon r^i(z) < 0,\ 1 \leqslant i \leqslant l\}. \tag{0.1}$$

The manifold

$$M = \{z \in U\colon r^i(z) = 0,\ 1 \leqslant i \leqslant l\} \tag{0.2}$$

is a *generic corner* of D if also $\partial r^1 \wedge \cdots \wedge \partial r^l \neq 0$ on M; i.e. the *complex* gradients of the r^i should be independent. M is a real submanifold of codimension l. The holomorphic tangent space $H_z(M)$, $z \in M$, is the vector space of all vectors of type $(1,0)$ annihilating the defining functions r^i at z. The condition means that H_z has complex codimension l. Any real submanifold M of $\mathbf{C}^n$ of codimension l satisfying this condition is called a *generic* real submanifold. If X and Y are local sections of $H(M)$ near z, the Levi form of M is defined by $(X, Y) \to L_z(X, Y) \equiv i[X, \overline{Y}]$, mod $H_z \oplus \overline{H}_z$. It is an hermitian bilinear form on H_z with values in $T_z \otimes C/H_z \oplus \overline{H}_z$, where T_z denotes the real tangent space of M at z. M is *nondegenerate* at z if the linear mapping $Y \to L_z(\cdot, Y)$ is injective on H_z.

We may now state the main result.

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THEOREM. *Let D be a domain in $\mathbf{C}^n$ with a generic real analytic corner M, and let M' be a generic analytic real submanifold with nondegenerate Levi form in $\mathbf{C}^m$. Suppose f is a holomorphic mapping from D to $\mathbf{C}^m$, which together with its first derivatives extends continuously to M, taking M into M'. If at some point z of M the differential of f induces a linear isomorphism of the holomorphic tangent spaces $H_z(M)$ and $H_{f(z)}(M')$, then f continues holomorphically to a full neighborhood of z in $\mathbf{C}^n$.*

REMARKS. (a) It is *not* required that the hypersurfaces $r^i = 0$ be analytic. It will be clear from the proof that the requirement that D have a corner along M is too strong. However, the points of M must be accessible via suitable cones lying in D. In one complex variable there are no generic corners other than smooth curves.

(b) The hypotheses imply that M and M' have the same holomorphic dimension. In case M is totally real it is not necessary to assume that the first derivatives of f extend continuously to M. The result then follows from the edge-of-the-wedge theorem after a change of coordinates.

(c) The theorem and its proof given below reduce to those given by H. Lewy in **[1]**, when $n = m$ and M and M' are both hypersurfaces. If, in addition, M and M' are both strongly pseudoconvex and f is biholomorphic, then the theorem, which is purely local, was proved in **[2]** under the assumption that f is Hölder continuous with exponent $\frac{1}{2} + \varepsilon$, $\varepsilon > 0$. See also **[5]** and Pinchuk **[3]**.

1. Reflection about a generic, nondegenerate submanifold. Let $M \subset \mathbf{C}^n$ be an analytic generic real submanifold of codimension l, $1 \leqslant l \leqslant n$. Choose a neighborhood U and local real analytic defining functions $r^i = r^i(z, \bar{z})$ for M as in (0.2). We may assume that the power series $r^i(z, \overline{w})$ converge for $z, w \in U$. Following **[5]** we define local nonsingular complex varieties Q_z of codimension l by $Q_z = \{w \in U: r^i(z, \overline{w}) = 0,\ 1 \leqslant i \leqslant l\}$. Because of the reality condition on the r^i, $w \in Q_z \Leftrightarrow z \in Q_w$. Given z near $z_0 \in M$ and a complex $(n - l)$-plane p nearly parallel to $H_{z_0}(M)$, we try to determine a "reflected" point $w \in Q_z$ by requiring $T_z Q_w = p$. If we set $q = T_w Q_z$, the problem is to set up an antiholomorphic involution $(z, p) \leftrightarrow (w, q)$ of pointed $(n - l)$-planes. p and q are elements of the complex Grassmanian $\mathrm{Gr}(n - l, n)$ of $(n - l)$-planes in $\mathbf{C}^n$. Gr has complex dimension $l(n - l)$, whereas the image of $w \to T_z Q_w$ has dimension at most $n - l$. Thus except for the hypersurface case $l = 1$, p and q must satisfy some consistency condition.

We consider also the complexification of M, $M^c = \{(z, w) \in U \times U: r^i(z, \overline{w}) = 0,\ 1 \leqslant i \leqslant l\}$. With z and $\eta = \overline{w}$ as variables, it is clear that M^c is a complex submanifold of $\mathbf{C}^{2n}$ of codimension l. There is a natural mapping π from M^c to $\mathbf{C}^n \times \mathrm{Gr}$ given by $\pi(z, w) = T_z Q_w$. π is holomorphic in z and antiholomorphic in w.

LEMMA. *The mapping π is an immersion at (z_0, z_0), $z_0 \in M$, if and only if the Levi form of M is nondegenerate at z_0.*

PROOF. This a matter of checking the definitions. We denote $\partial_a = \partial/\partial z^a$ and $\partial_{\bar{a}} = \partial/\partial \overline{w}^a$, $1 \leqslant a \leqslant n$. By a linear change of coordinates we may assume

$$(1.1)\qquad \begin{aligned} &\det\left(\partial_j r^i\right) \neq 0, \quad 1 \leqslant i, j \leqslant l, \\ &\partial_\alpha r^i = 0, \quad 1 \leqslant i \leqslant l,\ l < \alpha \leqslant n, \text{ at } (z, \overline{w}) = (z_0, \bar{z}_0). \end{aligned}$$

We define the operators

$$X_\alpha = \det\left[\begin{array}{c|c} \partial_\alpha & (\partial_j) \\ \hline (\partial_\alpha r^i) & (\partial_j r^i) \end{array}\right](z, \bar{w}), \tag{1.2}$$

for each α, $l < \alpha \leqslant n$, in which the $(l+1) \times (l+1)$ matrix is to be expanded across the top row. Clearly, the X_α are independent and annihilate the function $r^i(\cdot, \bar{w})$. They form a basis for the vectors of type $(1,0)$ tangent to Q_w at z. A basis $X_{\bar{\alpha}}$ for the $(0,1)$-tangent space of Q_w at z is given by (1.2) with ∂_α and ∂_j replaced by $\partial_{\bar{\alpha}}$ and $\partial_{\bar{j}}$, respectively. When $z = w \in M$, the X_α form a basis for $H_z(M)$. By (1.1) we may solve the equations $r^i(z, \bar{w}) = 0$ for z^i, $1 \leqslant i \leqslant l$, in terms of z^α, $l < \alpha \leqslant n$:

$$z^i = z^i(z^\alpha, \bar{w}), \quad p^i_\alpha = \frac{\partial z^i}{\partial z^\alpha}(z^\alpha, \bar{w}), \quad \partial_\alpha r^i + \sum_k p^k_\alpha \partial_k r^i = 0. \tag{1.3}$$

The p^i_α are coordinates for the plane $p = T_z Q_w$. It is clear that $\pi: (z, w) \to (z, p)$ is an immersion at (z_0, z_0) if and only if the $(n-l) \times l(n-l)$ matrix of derivatives (indexed by $(n-l)$ β's and $l(n-l)$ αj's)

$$\left(X_{\bar{\beta}} p^j_\alpha\right) \tag{1.4}$$

has rank $n - l$ when $z = w = z_0$. We want to show that this condition is equivalent to M having a nondegenerate Levi form at z_0. By Cartan's formula for exterior derivative the Levi form has the coordinate representation

$$\begin{aligned} L(X, Y) &= \left(i\partial r^1([X, \bar{Y}]), \dots, i\partial r^l([X, \bar{Y}])\right) \\ &= -i\left(\partial\bar{\partial} r^1(X, \bar{Y}), \dots, \partial\bar{\partial} r^l(X, \bar{Y})\right). \end{aligned}$$

We write

$$\partial\bar{\partial} r^j(X, \bar{Y}) = \sum \xi^a \bar{Y}\left[\partial_a r^j\right], \qquad X = \sum \xi^a \partial_a.$$

Since $\bar{Y}$ is a linear combination of the $X_{\bar{\alpha}}$, and $X_\alpha = \partial_\alpha$ at $(z_0, \bar{z}_0)$, the nondegeneracy of the Levi form is equivalent to the matrix $(X_{\bar{\beta}}[\partial_\alpha r^j])$ having rank $n - l$. If we differentiate the last equation in (1.3) with $X_{\bar{\beta}}$ and use the fact that $p^k_\alpha(z_0, \bar{z}_0) = 0$ and (1.1), we see that this matrix has the same rank as (1.4). $\square$

Let $\tilde{M}^c$ denote the image of π. Since π is holomorphic in $(z, \eta = \bar{w})$ and an immersion, $\tilde{M}^c$ is a (local) complex submanifold of $\mathbf{C}^n \times \mathrm{Gr}$ of dimension $2n - l$. By the reality condition on the r^i, M^c is invariant under the antiholomorphic involution $(z, \bar{w}) \to (w, \bar{z})$. This reflection induces a reflection on $\tilde{M}^c$ via π as follows. Given $(w, q) \in \tilde{M}^c$, $(w, q) = \pi(w, z)$ for a (locally) unique z. Use equation (1.3) with argument $(z, \bar{w})$ to define p. It is clear that the correspondence $(z, p) \to (w, q)$ is antiholomorphic and involutive.

2. Application to holomorphic mappings. In this section we prove the theorem. Let $D \cap U$ and M be given by (0.1) and (0.2), respectively. We first make a local coordinate change in a neighborhood of the particular point $z \in M$. After a translation and rotation we may assume that this $z = 0$ and that $T_0(M)$ is given by $y^j \equiv \mathrm{Im}\, z^j = 0$, $1 \leqslant j \leqslant l$, and that $H_0(M)$ is given by $z^j = 0$, $1 \leqslant j \leqslant l$. So z^α, $l < \alpha \leqslant n$, are coordinates on $H_0(M)$, and z^α, $x^i \equiv \mathrm{Re}\, z^j$, are coordinates on $T_0(M)$.

Locally, as a graph over $T_0(M)$, M is given by equations

$$\rho^j(z,\bar z) \equiv -y^j + h^j(z^\alpha, \bar z^\alpha, x^i) = 0, \qquad 1 \leqslant j \leqslant l, \tag{2.1}$$

in which the h^j are convergent power series about the origin which vanish together with their first derivatives when $z^\alpha = x^i = 0$. We define a *real* analytic local coordinate change T: $(\zeta^j, \zeta^\alpha) \to (z^j, z^\alpha)$, which is holomorphic in the ζ^j when the ζ^α are held constant, by

$$T\colon \begin{array}{l} z^\alpha = \zeta^\alpha \\ z^j = \zeta^j + ih^j(\zeta^\alpha, \bar\zeta^\alpha, \zeta^j). \end{array} \tag{2.2}$$

It is clear that the (real) Jacobian determinant does not vanish at the origin, and that $\operatorname{Im}\zeta^i = 0$ corresponds to M. For the functions r^i defining D the sets of covectors $\{\partial r^i\}$ and $\{\partial\rho^i\}$ in the z-coordinate system have the same linear span at points of M. Since $\bar\partial_\zeta z^j = 0$ at the origin of the ζ-system, $\{\partial_\zeta r^j\}$ and $\{d\zeta^j\}$ have the same span there. Since the first order approximatioin of D in the ζ system is the linear corner $\{d_\zeta r^j < 0\}$, it is clear that by a complex linear change of the ζ^j, $1 \leqslant j \leqslant l$, D can be made to contain the wedge $W^+ = (U_1 + iV^+) \times U_0$. Here U_0 is a neighborhood of $\zeta^\alpha = 0$ in the ζ^α-space, U_1 is a neighborhood of $\operatorname{Re}\zeta^j = 0$ in the $\operatorname{Re}\zeta^j$-space, and V^+ is the (truncated) cone $\operatorname{Im}\zeta^j > 0$, $1 \leqslant j \leqslant l$, in the $\operatorname{Im}\zeta^j$-space. We denote by V^- the (symmetrically truncated) cone $\operatorname{Im}\zeta^j < 0$, $1 \leqslant j \leqslant l$, and by W^- the corresponding wedge. If $z = (z^j, c^\alpha) = T(\zeta^j, c^\alpha)$ and $w = (w^j, c^\alpha) = T(\bar\zeta^j, c^\alpha)$, it is clear from (2.1) and (2.2) that $\rho^j(z, \bar w) = 0$, and that these equations characterize the reflection $\zeta^j \to \bar\zeta^j$.

Now we use the above to extend the mapping f. Let $\eta = (\eta^j, c^\alpha)$ be a point of W^-, $\zeta = (\bar\eta^j, c^\alpha)$, $w = T(\eta)$, $z = T(\zeta)$. Then $z \in D$ and $\rho^j(z, \bar w) = 0$; i.e. $z \in Q_w$. Let $p = T_z Q_w$, $z' = f(z)$, and $p' = df_z(p)$. As $\operatorname{Im}\eta^j \to 0$, it follows that $z \to z_0 \in M$, and $p \to H_{z_0}(M)$, for some z_0. Since f is C^1 and df is an isomorphism on $H(M)$, it follows that p' is an $(n-l)$-plane which approaches $p'_0 = H_{z'_0}(M')$, $z'_0 = f(z_0)$. In order to reflect (z', p') by the method of §1, we must show that $(z', p') \in \tilde M'^c$. Let $\Psi'(z', p') = 0$ denote (local) holomorphic functions defining the complex manifold $\tilde M'^c$. Since $(z'_0, p'_0) \in \tilde M'^c$, we see that $\Psi'(z', p') \to 0$ as $\operatorname{Im}\eta^j \to 0$. By construction $\Psi'(z', p')$ is an antiholomorphic function of η^j, for $\eta^\alpha = c^\alpha$ fixed, on W^-. We extend this function continuously to $W \equiv W^+ \cup W^-$ by setting it equal to 0 on W^+. By the edge-of-the-wedge theorem (see **[4]**) Ψ' continues holomorphically to a full neighborhood of $(0, c^\alpha)$ in the ζ^α-space. Since it vanishes on the real axis, it is identically zero. Hence, for $\operatorname{Im}\eta^j$ sufficiently small, uniformly in c^α, $(z', p') \in \tilde M'^c$, and the reflected point-plane (w', p') is defined and holomorphic in $\eta^j \in W^-$, for each fixed c^α.

Thus $\eta \to w'(\eta^j, c^\alpha)$ gives a continuous extension $\tilde f$ of $f \circ T$ to W. We now get an extension $\tilde F$ of $\tilde f$ to a full neighborhood of 0 in C^n. This is given by the following integral, formula (6), §4, of **[4]**,

$$2\pi \tilde F(\zeta^j, c^\alpha) = \int_{-\pi}^{\pi} \tilde f\big(\Phi(\zeta^i, e^{i\theta}), c^\alpha\big)\, d\theta.$$

$\tilde F$ is continuous in (ζ^j, c^α) since $\tilde f$ is continuous on W. It is also holomorphic on each plane $\zeta^\alpha = c^\alpha$. Set $F = \tilde F \circ T^{-1}$. F continuously extends f to a neighborhood of the original point of M and is holomorphic on each l-plane $\zeta^\alpha = c^\alpha$. As in **[1]** (or see **[2]**), we can argue that F is holomorphic as follows. Let $I_c(F)$ be the complex line integral of F about a small loop c in the complex z^α-plane. $I_c(F)$ is holomorphic in z^j and vanishes for the open set of z^j for which $(z^j, c) \subset D$. Hence, $I_c(F) \equiv 0$, and F is holomorphic in z^α by Morera's theorem.

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