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DISTANCE ESTIMATES FOR VON NEUMANN ALGEBRAS

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ABSTRACT. It is shown that for certain von Neumann algebras $\mathcal{A}$, there is a constant C such that

$$\operatorname{dist}(T, \mathcal{A}) \leqslant C \sup_{P \in \operatorname{lat} \mathcal{A}} \| P^{\perp} T P \| \quad \text{for all } T \text{ in } \mathcal{B}(\mathcal{H}).$$

1. Introduction. Throughout this paper, $\mathcal{H}$ denotes a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. For any subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, let lat $\mathcal{A}$ denote the lattice of orthogonal projections P invariant for $\mathcal{A}$. That is, $P^{\perp} A P = 0$ for all A in $\mathcal{A}$, where $P^{\perp} = I - P$. $\mathcal{A}$ is said to be reflexive if every operator B satisfying $P^{\perp} B P = 0$ for all P in lat $\mathcal{A}$ belongs to $\mathcal{A}$.

Let $\mathcal{A}$ be a reflexive algebra and T an arbitrary operator in $\mathcal{B}(\mathcal{H})$. It is easy to see that

$$\operatorname{dist}(T, \mathcal{A}) \geqslant \sup_{P \in \operatorname{lat} \mathcal{A}} \| P^{\perp} T P \|. \tag{1}$$

Arveson [1] proved that if $\mathcal{A}$ is a nest algebra, then equality actually occurs in (1). Davidson [3] has referred to Choi's example which shows that equality fails to hold even if $\mathcal{H}$ is finite dimensional and $\mathcal{A}$ is a m.a.s.a. He asked: if $\mathcal{A}$ is reflexive and lat $\mathcal{A}$ is commutative, then is there a constant C such that

$$\operatorname{dist}(T, \mathcal{A}) \leqslant C \sup_{P \in \operatorname{lat} \mathcal{A}} \| P^{\perp} T P \| \tag{2}$$

for all T in $\mathcal{B}(\mathcal{H})$?

In this paper, we shall prove that (2) holds (with $C = 2$) if $\mathcal{A}$ is a von Neumann algebra such that either $\mathcal{A}$ or $\mathcal{A}'$ is abelian. Note that if $\mathcal{A}'$ is commutative, then so is lat $\mathcal{A}$. Also if $\mathcal{A}$ is a weakly closed unital algebra of normal operators, then (2) holds with $C = 3$.

In [5], Johnson conjectured that for a von Neumann algebra $\mathcal{A}$ there is a positive constant K such that for all T in $\mathcal{B}(\mathcal{H})$,

$$\operatorname{dist}(T, \mathcal{A}) \leqslant K \| \Delta_T |_{\mathcal{A}'} \| \tag{3}$$

where Δ_T is the derivation $\Delta_T(S) = ST - TS$. Christensen [2] proved that (3) holds for a very large class of von Neumann algebras. We will show that (2) and (3) are equivalent.

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2. Distance estimates and derivations.

THEOREM 2.1. *Let $\mathcal{A}$ be a von Neumann algebra and let T belong to $\mathcal{B}(\mathcal{H})$.*
(i) *If* $\operatorname{dist}(T, \mathcal{A}) \leqslant K \|\Delta_T|_{\mathcal{A}'}\|$, *then*

$$\operatorname{dist}(T, \mathcal{A}) \leqslant 4K \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|.$$

(ii) *If* $\operatorname{dist}(T, \mathcal{A}) \leqslant C \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|$, *then*

$$\operatorname{dist}(T, \mathcal{A}) \leqslant \frac{C}{2} \|\Delta_T|_{\mathcal{A}'}\|.$$

PROOF. Since lat $\mathcal{A}$ is complemented, we have

$$\begin{aligned} 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\| &= 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \max\{\|P^{\perp} TP\|, \|PTP^{\perp}\|\} \\ &= 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \|PT - TP\| \\ &= \sup_{P \in \operatorname{lat} \mathcal{A}} \|(2P - I)T - T(2P - I)\| \leqslant \|\Delta_T|_{\mathcal{A}'}\|. \end{aligned}$$

This proves (ii).

Now suppose $\sup_{P \in \operatorname{lat} \mathcal{A}} \|PT - TP\| = \delta$. Then for P in lat $\mathcal{A}$, $\|\Delta_T(2P - I)\| \leqslant 2\delta$. Let $\mathfrak{M}$ be the real vector space of all Hermitian operators in $\mathcal{A}'$. By the Krein-Milman theorem, the unit ball of $\mathfrak{M}$ is the weakly closed convex hull of its extreme points. But these extreme points are precisely $(2P - I)$ for projections P in $\mathcal{A}'$, namely lat $\mathcal{A}$. Thus $\|\Delta_T|_{\mathfrak{M}}\| \leqslant 2\delta$. If B in $\mathcal{A}'$ has $\|B\| \leqslant 1$, then write $B = A_1 + iA_2$ where A_i are Hermitian and $\|A_i\| \leqslant 1$. Then $\|\Delta_T(B)\| \leqslant \|\Delta_T(A_1)\| + \|\Delta_T(A_2)\| \leqslant 4\delta$, which proves (i). □

3. Abelian von Neumann algebras. In [2], it is established that if $\mathcal{A}'$ is abelian,

$$\operatorname{dist}(T, \mathcal{A}) \leqslant \|\Delta_T|_{\mathcal{A}'}\|.$$

By Theorem 2.1, we conclude that (1) holds with $C = 4$. In fact, we have

LEMMA 3.1. *If $\mathcal{A}$ is a von Neumann algebra with abelian commutant, then for every T in $\mathcal{B}(\mathcal{H})$,*

$$\operatorname{dist}(T, \mathcal{A}) \leqslant 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|.$$

PROOF. Let $\delta = \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|$. Let G be the group of unitaries generated by $\{2P - I\colon P \in \operatorname{lat} \mathcal{A}\}$. Then for every element U in G, $\|UT - TU\| \leqslant 2\delta$, whence $\|T - U^{-1}TU\| \leqslant 2\delta$. Since G is abelian, it has an invariant mean m. Let $f(U) = U^{-1}TU$, and define $T_0 = m(f)$ following the method of [6]. Since T_0 is in the weakly closed convex hull of $\{U^{-1}TU\colon U \in G\}$, we have $\|T - T_0\| \leqslant 2\delta$. But the invariance of m shows that T_0 belongs to $G' = (\mathcal{A}')' = \mathcal{A}$. Thus $\operatorname{dist}(T, \mathcal{A}) \leqslant 2\delta$. □

LEMMA 3.2. *Let ϕ be a functional on $\mathcal{B}(\mathcal{H})$ continuous in the weak operator topology such that $\phi(I) = 0$. Then the kernel of ϕ contains a m.a.s.a.*

PROOF. There is a finite rank operator A such that $\phi(T) = \operatorname{tr}(AT)$ for all T in $\mathcal{B}(\mathcal{H})$. It suffices to find an orthonormal basis $\{f_i\}$ for $\mathcal{H}$ such that $(Af_i, f_i) = 0$ for all i. For then, we take our m.a.s.a. to be all operators which are diagonal with respect to this basis.

The numerical range of an operator B, namely $W(B) = \{(Bx, x): \|x\| = 1\}$, is always convex and nonempty. When B acts on a space of finite dimension n, we have

$$\frac{1}{n}\sum_{i=1}^{n}(Be_i, e_i) = \frac{1}{n}\operatorname{tr} B.$$

So if $\operatorname{tr} B = 0$, then 0 belongs to $W(B)$ and thus there is a unit vector f such that $(Bf, f) = 0$. On the complement $\mathfrak{M}$ of $\operatorname{span}\{f\}$, we have

$$0 = \operatorname{tr} B = (Bf, f) + \operatorname{tr}(B|_{\mathfrak{M}}) = \operatorname{tr}(B|_{\mathfrak{M}}).$$

By induction, there is an orthonormal basis $\{f_i, 1 \leq i \leq n\}$ with $(Bf_i, f_i) = 0$.

Choose a finite dimensional subspace $\mathfrak{N}$ which reduces A and $A|_{\mathfrak{N}^\perp} = 0$. Apply the previous paragraph to $A|_{\mathfrak{N}}$ and complete the orthonormal set with an arbitrary basis for $\mathfrak{N}^\perp$. $\square$

LEMMA 3.3. *Let $\mathcal{A}$ be an abelian von Neumann algebra on $\mathcal{H}$, and let ϕ be a weak operator continuous functional on $\mathcal{B}(\mathcal{H})$ which annihilates $\mathcal{A}$. Then there is a m.a.s.a. $\mathfrak{M}$ containing $\mathcal{A}$ in the kernel of ϕ.*

PROOF. We shall use the direct integral decomposition of $\mathcal{A}$ [**8**, p. 19]. There is a measure space (Z, μ) such that $\mathcal{H} = \int^{\oplus}\mathcal{H}(\zeta)\,d\mu(\zeta)$ and $\mathcal{A}$ is the algebra of all operators $T = \int^{\oplus} T(\zeta)\,d\mu(\zeta)$ where $T(\zeta)$ is a scalar multiple of the identity $I_{\mathcal{H}(\zeta)}$. We can write ϕ in the form $\phi(T) = \sum_{i=1}^{n}(Tx_i, y_i)$, and $x_i = \int^{\oplus} x_i(\zeta)\,d\mu(\zeta)$ and $y_i = \int^{\oplus} y_i(\zeta)\,d\mu(\zeta)$.

Define $f(\zeta) = \sum_{i=1}^{n}(x_i(\zeta), y_i(\zeta))$. Then $f(\zeta)$ belongs to $L^1(Z, \mu)$. If $g(\zeta)$ is a bounded measurable function on (Z, μ), then

$$0 = \phi\left(\int^{\oplus} g(\zeta) I_{\mathcal{H}(\zeta)}\,d\mu(\zeta)\right) = \int f(\zeta) g(\zeta)\,d\mu(\zeta).$$

Hence $f(\zeta) = 0$ a.e.

Let ϕ_ζ be the weak operator continuous functional on $\mathcal{B}(\mathcal{H}(\zeta))$ given by $\phi_\zeta(S) = \sum_{i=1}^{n}(Sx_i(\zeta), y_i(\zeta))$. Then $\phi_\zeta(I) = 0$ for almost all ζ. So by Lemma 3.2, there is a m.a.s.a. $\mathfrak{M}(\zeta)$ in the kernel of ϕ_ζ. Let $\mathfrak{M} = \int^{\oplus}\mathfrak{M}(\zeta)\,d\mu(\zeta)$. It is easy to verify that $\mathfrak{M}$ is a m.a.s.a. in $\mathcal{B}(\mathcal{H})$ and $\phi(\mathfrak{M}) = 0$. $\square$

LEMMA 3.4. *If $\mathcal{A}$ is an abelian von Neumann algebra, and T belongs to $\mathcal{B}(\mathcal{H})$, then*

$$\operatorname{dist}(T, \mathcal{A}) = \sup \operatorname{dist}(T, \mathfrak{M}) \tag{4}$$

where the sup is taken over all m.a.s.a.'s $\mathfrak{M}$ containing $\mathcal{A}$.

PROOF. We will prove the nontrivial inequality $\operatorname{dist}(T, \mathcal{A}) \geq \sup \operatorname{dist}(T, \mathfrak{M})$. Let δ denote the right-hand side of (4), and fix $\varepsilon > 0$. For each m.a.s.a. $\mathfrak{M}$ containing $\mathcal{A}$, choose $T_{\mathfrak{M}}$ in $\mathfrak{M}$ with $\|T - T_{\mathfrak{M}}\| < \delta + \varepsilon$. Let $\mathcal{K}$ denote the weak operator closed convex hull of $\{T_{\mathfrak{M}}\}$. If $\mathcal{K}$ were disjoint from $\mathcal{A}$, then by the Hahn-Banach theorem, there is a weak operator continuous linear functional ϕ on $\mathcal{B}(\mathcal{H})$ which annihilates $\mathcal{A}$ but is nonzero on all of $\mathcal{K}$. But by Lemma 3.3, there is a m.a.s.a. $\mathfrak{M}_0$ containing $\mathcal{A}$ in the kernel of ϕ. In particular, $\phi(T_{\mathfrak{M}_0}) = 0$ which is a contradiction. Hence $\mathcal{K}$ meets $\mathcal{A}$. Let A belong to the intersection $\mathcal{K} \cap \mathcal{A}$. Then $\|T - A\| \leq \delta + \varepsilon$. Since ε was arbitrary, $\operatorname{dist}(T, \mathcal{A}) \leq \delta$. $\square$

THEOREM 3.5. *If A is a von Neumann algebra such that $\mathcal{A}$ or $\mathcal{A}'$ is abelian, then for all T in $\mathcal{B}(\mathcal{H})$,*

$$\operatorname{dist}(T, \mathcal{A}) \leqslant 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|.$$

PROOF. Lemma 3.1 suffices if $\mathcal{A}'$ is abelian. If $\mathcal{A}$ is abelian and $\mathfrak{M}$ is a m.a.s.a. containing $\mathcal{A}$, then lat $\mathfrak{M} \subseteq$ lat $\mathcal{A}$, so by Lemma 3.4,

$$\begin{aligned}\operatorname{dist}(T, \mathcal{A}) &= \sup_{\mathfrak{M}} \operatorname{dist}(T, \mathfrak{M}) \\ &\leqslant \sup_{\mathfrak{M}} 2 \sup_{P \in \operatorname{lat} \mathfrak{M}} \|P^{\perp} TP\| \leqslant 2 \sup_{P \in \operatorname{lat} \mathcal{A}} \|P^{\perp} TP\|. \quad \square\end{aligned}$$

We conclude this section with a distance estimate for (possibly non-self-adjoint) algebras of normal operators.

THEOREM 3.6. *If $\mathcal{B}$ is a unital weakly closed algebra of normal operators, then for all T in $\mathcal{B}(\mathcal{H})$,*

$$\operatorname{dist}(T, \mathcal{B}) \leqslant 3 \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} TP\|.$$

PROOF. By [7, Lemma 9.20], $\mathcal{B}$ is abelian. Let $\mathfrak{M}$ be a m.a.s.a. containing $\mathcal{B}$. We may assume that $\|T\| = 1$. Let T_0 be the operator produced in the proof of Lemma 3.1. If U is a unitary in $\mathfrak{M}$, and P belongs to lat $\mathcal{B}$, let $Q = UPU^*$. For B in $\mathcal{B}$,

$$Q^{\perp} BQ = UP^{\perp}(U^*BU)PU^* = U(P^{\perp} BP)U^* = 0.$$

Thus Q belongs to lat $\mathcal{B}$. Now

$$\|P^{\perp}(U^*TU)P\| = \|(UP^{\perp} U^*)T(UPU^*)\| = \|Q^{\perp} TQ\|.$$

Since T_0 belongs to the weakly closed convex hull of $\{U^*TU\}$,

$$\sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} T_0 P\| \leqslant \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} TP\|. \tag{5}$$

Also, by Lemma 3.1,

$$\|T - T_0\| \leqslant 2 \sup_{P \in \operatorname{lat} \mathfrak{M}} \|P^{\perp} TP\| \leqslant 2 \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} TP\|.$$

We can complete the proof by proving that $\operatorname{dist}(T_0, \mathcal{B}) \leqslant \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} T_0 P\|$. Now $\operatorname{dist}(T_0, \mathcal{B}) = \sup |\phi(T_0)|$ where ϕ runs over all weak * continuous functionals on $\mathfrak{M}$ of norm one which vanish on $\mathcal{B}$. Let $\varepsilon > 0$, and choose such a functional ϕ with $\operatorname{dist}(T_0, \mathcal{B}) < |\phi(T_0)| + \varepsilon$. Since $\mathfrak{M}$ is maximal abelian, there are vectors x and y such that $\phi(M) = (Mx, y)$ for all M in $\mathfrak{M}$ and $\|x\| \|y\| \leqslant 1 + \varepsilon$. Let P_0 be the orthogonal projection onto the closed span of $\mathcal{B}x$. Clearly, P_0 belongs to lat $\mathcal{B}$, $P_0 x = x$, and $P_0^{\perp} y = y$. So

$$\begin{aligned}\operatorname{dist}(T_0, \mathcal{B}) &\leqslant |(T_0 x, y)| + \varepsilon = |(P_0^{\perp} T_0 P_0 x, y)| + \varepsilon \\ &\leqslant \|P_0^{\perp} T_0 P_0\| \|x\| \|y\| + \varepsilon \leqslant (1 + \varepsilon) \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} T_0 P\| + \varepsilon.\end{aligned}$$

Thus, $\operatorname{dist}(T_0, \mathcal{B}) \leqslant \sup_{P \in \operatorname{lat} \mathcal{B}} \|P^{\perp} T_0 P\|$, and the theorem is proven. $\square$

4. Extensions of derivations. Let $\mathcal{A}$ be an abelian von Neumann algebra. A derivation from A to $\mathcal{B}(\mathcal{H})$ is a linear map Δ satisfying $\Delta(AB) = (\Delta A)B + A(\Delta B)$. It is well known that every bounded derivation from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ can be extended to an (inner) derivation of $\mathcal{B}(\mathcal{H})$. Here we give a slight strengthening of this using a technique developed in [1] and [3].

THEOREM 4.1. *Let $\mathcal{P}$ be the set of projections in an abelian von Neumann algebra $\mathcal{A}$. Suppose Δ is a map of $\mathcal{P}$ into $\mathcal{B}(\mathcal{H})$ satisfying*

(i) $\Delta(P+Q) = \Delta P + \Delta Q$ when $PQ = 0$,

(ii) $\Delta(PQ) = (\Delta P)Q + P(\Delta Q)$ and

(iii) $\|\Delta P\| \leq M$ for all P, Q in $\mathcal{P}$.

Then there is a T in $\mathcal{B}(\mathcal{H})$ with $\|T\| \leq 2M$ such that $\Delta = \Delta_T | \mathcal{P}$.

PROOF. Let $\mathcal{L} = \{P_j, 1 \leq j \leq n\}$ be a finite subset of $\mathcal{P}$ with $\sum_{j=1}^n P_j = I$. Define $T_{\mathcal{L}} = \sum_{i \neq j} P_j \Delta P_i$. The standard argument shows that $\Delta I = 0$ and thus $\sum_{i=1}^n \Delta P_i = 0$. Compute

$$\begin{aligned} T_{\mathcal{L}} P_m - P_m T_{\mathcal{L}} &= \sum_{i \neq j} P_j (\Delta P_i) P_m - P_m \sum_{i \neq j} P_j \Delta P_i \\ &= -\sum_{i \neq j} (\Delta P_j) P_i P_m - P_m \sum_{i \neq m} \Delta P_i \\ &= -\Big(\sum_{j \neq m} \Delta P_j \Big) P_m - P_m \Big(\sum_{i \neq m} \Delta P_i \Big) \\ &= (\Delta P_m) P_m + P_m (\Delta P_m) = \Delta P_m. \end{aligned}$$

Thus, for every projection P in $\mathcal{L}''$, $T_{\mathcal{L}} P - P T_{\mathcal{L}} = \Delta P$. By Lemma 3.1,

$$\operatorname{dist}(T_{\mathcal{L}}, \mathcal{L}') \leq 2 \sup_{P \in \mathcal{L}''} \|T_{\mathcal{L}} P - P T_{\mathcal{L}}\| \leq 2 \sup_{P \in \mathcal{L}''} \|\Delta P\| \leq 2M.$$

Choose an $A_{\mathcal{L}}$ in $\mathcal{L}'$ with $\|T_{\mathcal{L}} - A_{\mathcal{L}}\| \leq 2M + 1/n$. Set $S_{\mathcal{L}} = T_{\mathcal{L}} - A_{\mathcal{L}}$. Then $S_{\mathcal{L}} P - P S_{\mathcal{L}} = \Delta P$ for every P in $\mathcal{L}''$.

Since all finite subsets of $\mathcal{P}$ with sum I form a directed set and the ball of radius $2M + 1$ is weakly compact, the net $\{S_{\mathcal{L}}\}$ has a limit point T. Clearly, $\|T\| \leq 2M$ and $TP - PT = \Delta P$ for all P in $\mathcal{P}$.

It has come to our attention that Theorem 2.1 and Lemma 3.1 have been proven independently by Gilfeather and Larson [4], using similar methods.

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