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A ZERO-ONE, BOREL PROBABILITY WHICH ADMITS OF NO COUNTABLY ADDITIVE EXTENSIONS

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ABSTRACT. There is a subsigma-field of the Borel subsets $\mathcal{B}$ of the unit interval which supports a countably additive, two-valued, probability which cannot be extended to $\mathcal{B}$ so as to remain countably additive.

This note notes the existence of a countably additive probability P defined for a sigma-field of Borel subsets of $\mathbf{R}^\infty$ which satisfies this peculiar property: If $\omega = (\omega_1, \omega_2, \dots) \in \mathbf{R}^\infty$ is P-distributed, then, for *every* real number t, with P-probability 1, $\omega_n = t$ for some n.

Formally, let E_t be the set of $\omega \in \mathbf{R}^\infty$ such that $\omega_n = t$ for at least one positive integer n, and let $\mathcal{U}$ be the sigma-field generated by the family $\{E_t, t \in \mathbf{R}\}$.

PROPOSITION 1. *There is one, and only one, countably additive probability P defined on $\mathcal{U}$ such that $P(E_t) = 1$ for all t.*

PROOF. As is easily verified, there is one, and only one, finitely additive probability P_1, defined on the field $\mathcal{F}$ generated by the E_t such that $P_1(E_t) = 1$ for all t. Let $F_i \in \mathcal{F}$, $F_{i+1} \subset F_i$, $1 \le i < \infty$ with $\inf P_1(F_i) = \epsilon > 0$. To see that P_1 is countably additive, it is only necessary to see that $\bigcap F_i$ is nonempty. To this end, note first that $\epsilon = 1$, so $P(F_i) = 1$ for all i. Now call a sequence $G_1, G_2, \dots$ a *selection* if, for each n, G_n is $E_{t(n)}$, where $t(1), t(2), \dots$ is a sequence of indices, and then verify: (a) if $G_1, G_2, \dots$ is any selection, then $\bigcap G_n$ is nonempty; and (b) there is a selection $G_1, G_2, \dots$ such that $\bigcap G_n \subset \bigcap F_i$. □

For each real number t and positive integer n, let $E_{t,n}$ be the set of all infinite sequences whose nth coordinate has the value t, and let $\mathcal{V}$ be the sigma-field generated by the collection of all $E_{t,n}$. Since, for each t, E_t is the union of the $E_{t,n}$, $E_t \in \mathcal{V}$, so $\mathcal{U} \subset \mathcal{V}$.

A probability Q on a sigma-field is *purely finitely additive* if there is a denumerable collection of elements of the sigma-field, each of Q-probability zero, whose union has Q-probability one.

LEMMA 1. *Let Q be a probability defined on $\mathcal{V}$, or on any $\mathcal{W}$ which includes $\mathcal{V}$, and suppose that $Q(E_t) = 1$ for a nondenumerable set of t. Then Q is purely finitely additive.*

PROOF. For fixed n, the set of $E_{t,n}$ are disjoint and, therefore, the set, T_n, of t such that $E_{t,n}$ has positive Q-probability is countable. Consequently, the union, T, of the T_n is also countable. So there certainly is a t not in T. For such a t, $E_{t,n}$ has Q-probability zero for all n, but their set-theoretic union, namely E_t, has Q-probability 1. □

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THEOREM 1. *Let I be any complete, separable, metrizable space with a continuum of points. Then there is a $(\mathcal{U}, P)$ where* : (a) *$\mathcal{U}$ is a subsigma-field of the Borel subsets $\mathcal{B}$ of I*; (b) *P is a countably additive, two-valued probability defined on $\mathcal{U}$*; (c) *every probability on $\mathcal{B}$ which agrees with P on $\mathcal{U}$ is purely finitely additive.*

PROOF OF THEOREM 1. For $I = \mathbf{R}^\infty$, Proposition 1, together with Lemma 1, implies the conclusion. The conclusion for any I follows for, as is well known, $(I, \mathcal{B}(I))$ is isomorphic to $(\mathbf{R}^\infty, \mathcal{B}(\mathbf{R}^\infty))$. □

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