PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

ON PROXIMINALITY IN $L_1(T \times S)$

S. M. HOLLAND[1], W. A. LIGHT AND L. J. SULLEY

ABSTRACT. It is proved that if G and H are finite-dimensional subspaces of $L_1(S)$ and $L_1(T)$ respectively then each element of $L_1(T \times S)$ has a best approximation in the subspace $L_1(T) \otimes G + H \otimes L_1(S)$.

1. Introduction. Let W be a subspace of a normed linear space X. W is said to be proximinal in X if to each f in X there corresponds a closest point w^* in W; that is, a point w^* in W such that $\| f - w^* \| \leqslant \| f - w \|$ for all w in W.

We consider two finite measure spaces (T, Θ, μ) and (S, Φ, ν). The product space $T \times S$ becomes a measure space $(T \times S, \Omega, \sigma)$ by means of a standard construction. Let $G = [g_1, g_2, \ldots, g_n]$ by a finite-dimensional subspace of $L_1(S)$ and $H = [h_1, h_2, \ldots, h_m]$ be a similar subspace of $L_1(T)$. Set $U = L_1(T) \otimes G$ and $V = H \otimes L_1(S)$. A typical element u of U has the form $u(t, s) = \sum_1^n x_i(t) g_i(s)$ where $x_i \in L_1(T)$. We shall take X to be $L_1(T \times S)$ and W to be $U + V$.

It is known from [3] and earlier work in [1] that if f is essentially bounded on $T \times S$, then it has a closest point in W (distance being measured in the L_1-norm). We shall establish the more general result.

THEOREM. *The subspace $W = L_1(T) \otimes G + H \otimes L_1(S)$ is proximinal in $L_1(T \times S)$.*

2. Preliminaries. In this section we present the three strands which will combine to prove the main result.

Unadorned norm symbols will denote the L_1-norm on $T \times S$, whereas subscripts will be used to denote L_1-norms on T and S. For example,

$$\|f\| = \iint_{T \times S} |f(t, s)| d\mu\, d\nu, \qquad f \in L_1(T \times S),$$

while

$$\|v\|_S = \int_S |v(s)| d\nu, \qquad v \in L_1(S).$$

The first strand is the Dunford-Pettis theorem [2, p. 294].

THEOREM A (DUNFORD-PETTIS). *A set K in $L_1(T \times S)$ is weakly relatively sequentially compact if and only if it is bounded and*

$$\lim_{\sigma(E) \to 0} \int_E f d\sigma = 0 \quad \textit{uniformly for } f \textit{ in } K.$$

Received by the editors November 13, 1981.
1980 *Mathematics Subject Classification*. Primary 41A65, 41A50, 41A44.
[1] This author was supported by the Science and Engineering Research Council of Great Britain.

0002-9939/82/0000-0326/$01.50

By the Eberlein-Smulian theorem [2, p. 430], this condition is also necessary and sufficient for weak relative compactness in $L_1(T \times S)$. The sufficiency in Theorem A only holds good since $(T \times S, \Omega, \sigma)$ is a *finite* measure space.

The second result comes from [3]. It is a summary of the construction carried out in the proof of Theorem 1 therein. We adopt the notation f_t, f^s where $f_t(s) = f(t, s) = f^s(t)$. By the Fubini theorem, if $f \in L_1(T \times S)$, then $f_t \in L_1(S)$ for almost all t in T and $f^s \in L_1(T)$ for almost all s in S.

LEMMA B. *To each f in $L_1(T \times S)$ there corresponds a closest point u in U such that u_t is a closest point in G to f_t for almost all t in T.*

Finally, our third tool is the following elementary result:

LEMMA C. *There exists a function g in $L_1(S)$ such that, for each u in U,*
(i) $|u(t, s)| \leqslant g(s)\|u_t\|_S$,
(ii) $\|u_s\|_T \leqslant g(s)\|u\|$,
for almost all t in T and s in S.

PROOF. Set $d_j^{-1} = \inf_{c_i \in R} \|\Sigma_{i \neq j} c_i g_i + g_j\|_S$. Since the g_i are linearly independent, we have $d_j^{-1} > 0$ for $j = 1, 2, \ldots, n$. Let $u = \Sigma_1^n x_i g_i$ in U and let $T_j = \{t \in T: x_j(t) \neq 0\}$. Then for t in T_j we have

$$\left\| \sum_1^n x_i(t) g_i \right\|_S = |x_j(t)| \left\| \sum_{i=1}^n \frac{x_i(t)}{x_j(t)} g_i \right\|_S \geqslant |x_j(t)| d_j^{-1}.$$

So for all t in T, $|x_j(t)| \leqslant d_j \|u_t\|_S$. Now

$$|u(t, s)| \leqslant \sum_1^n |x_i(t)|\,|g_i(s)| \leqslant \|u_t\|_S \sum_1^n d_i |g_i(s)|.$$

Choosing $g = \Sigma_1^n d_i |g_i|$, (i) is proved. To obtain (ii),

$$\|u_s\|_T = \int_T |u(t, s)| d\mu \leqslant \int_T g(s) \|u_t\|_S \, d\mu = g(s)\|u\|.$$

3. Proof of the theorem. In accordance with Lemma B, we define mappings (which are termed metric selections) A_U: $L_1(T \times S) \to U$ and A_V: $L_1(T \times S) \to V$ such that $(A_U f)_t$ is a closest point to f_t for almost all t in T and $(A_V f)^s$ is a closest point to f^s for almost all s in S. Throughout the rest of this section f will be a fixed member of $L_1(T \times S)$. We can now define mappings B_U: $V \to U$ and B_V: $U \to V$ by $B_U v = A_U(f - v)$ and $B_V u = A_V(f - u)$.

THEOREM D. *The mappings B_U and B_V are weakly compact.*

PROOF. We shall only verify that B_U is weakly compact, the case of B_V being similar.

Let $K = \{v \in V: \|v\| \leqslant k\}$. We shall show that $B_U K$ is weakly relatively compact in $L_1(T \times S)$. Since $\|(f - v)_t - (A_U(f - v))_t\|_S \leqslant \|(f - v)_t\|_S$ for almost all t in T, we have

$$\|(B_U v)_t\|_S = \|(A_U(f - v))_t\|_S \leqslant 2\|(f - v)_t\|_S \leqslant 2\|f_t\|_S + 2\|v_t\|_S$$

for almost all t in T. By Lemma C(i)

$$|B_U v(t,s)| \leqslant g(s)\|(B_U v)_t\|_S \leqslant 2g(s)\big(\|f_t\|_S + \|v_t\|_S\big).$$

Now applying Lemma C(ii) to V instead of U, there is an h in $L_1(T)$ such that $\|v_t\|_S \leqslant h(t)\|v\|$ for all v in V. Then

$$\begin{aligned} |B_U v(t,s)| &\leqslant 2g(s)\big(\|f_t\|_S + h(t)\|v\|\big) \\ &\leqslant 2g(s)\big(\|f_t\|_S + k(h(t))\big) \quad \text{for } v \text{ in } K. \end{aligned}$$

The right-hand side of this inequality is a member of $L_1(T\times S)$ which is independent of v in K. Hence if Q is a measurable set in $T\times S$,

$$\int_Q |B_U v|\,d\sigma \to 0 \text{ as } \sigma(Q)\to 0 \quad \text{uniformly over } v \text{ in } K.$$

By the Dunford-Pettis theorem (Theorem A), $B_U K$ is weakly relatively compact.

Theorem D is the essential tool used to establish the proximinality of $W = U + V$ in $L_1(T\times S)$. However, a necessary condition for W to be proximinal is that it be closed. We need to use the fact that W is closed. This result was given in [3] and we reproduce it here on account of its brevity.

LEMMA E. *The subspace $W = U + V$ is closed in $L_1(T\times S)$. There is a constant β such that each element w of W has a representation $w = u + v$ with $u\in U$, $v\in V$ and $\|u\| + \|v\| \leqslant \beta\|w\|$.*

PROOF. Let biorthonormal bases $\{g_i\}_1^n$, $\{\phi_i\}_1^n$ be chosen for G, G^* and $\{h_i\}_1^m$, $\{\psi_i\}_1^m$ for H, H^*. Then define

$$(Pf)(t,s) = \sum_{i=1}^{n} \langle f_t, \phi_i\rangle g_i(s), \qquad f\in L_1(T\times S),$$

$$(Qf)(t,s) = \sum_{i=1}^{m} \langle f^s, \psi_i\rangle h_i(t), \qquad f\in L_1(T\times s).$$

These are (bounded, linear) projections of $L_1(T\times S)$ onto U and V respectively. It is easily verified that $PQ = QP$. By well-known results, $P + Q - PQ$ is a projection of $L_1(T\times S)$ onto W. The latter is therefore closed. Now given w in W, we set $u = Pw - PQw$ and $v = Qw$, when $w = u + v$ is the required representation of w.

To prove the proximinality of W in $L_1(T\times S)$, let f be any element of $L_1(T\times S)$. Let (w_n) be a minimising sequence for f; i.e. $\|f - w_n\| \to \operatorname{dist}(f, W)$. We can assume without loss of generality that $\|w_n\| \leqslant 2\|f\|$ for all n. Then by Lemma E, we can write $w_n = u_n + v_n$ where (u_n) and (v_n) are bounded sequences in U and V respectively. Define $v_n^* = B_V u_n$ and $u_n^* = B_U v_n^*$.

$$\|f - u_n^* - v_n^*\| = \|f - v_n^* - A_U(f - v_n^*)\| \leqslant \|f - v_n^* - u_n\|$$

since $A_U(f - v_n^*)$ is a closest point in U to $f - v_n^*$. Similarly,

$$\|f - u_n^* - v_n^*\| \leqslant \|f - v_n^* - u_n\| = \|f - u_n - A_V(f - u_n)\| \leqslant \|f - u_n - v_n\|.$$

Thus if $w_n^* = u_n^* + v_n^*$, (w_n^*) is a minimising sequence for f. By Theorem D the set $\{w_n^*\}$ is weakly relatively compact. Furthermore, W is closed by Lemma E and so (w_n^*) has a weak cluster point w in W. Since the norm is weakly lower semicontinuous, this point w is a closest point to f in W.

References

1. E. W. Cheney, J. H. McCabe, W. A. Light and G. Phillips, *The approximation of bivariate functions by sums of univariate ones using the L_1-metric*, Center for Numerical Analysis Technical Report, University of Texas, 1979.

2. N. Dunford and J. T. Schwartz, *Linear operators*, Part 1, Interscience, New York, 1959.

3. W. A. Light and E. W. Cheney, *Some best-approximation theorems in tensor-product spaces*, Proc. Cambridge Philos. Soc. **89** (1981), 385–390.

4. J. R. Respess and E. W. Cheney, *Best approximation problems in tensor-product spaces*, Pacific J. Math. (to appear).

Department of Mathematics, University of Lancaster, Bailrigg, Lancaster, United Kingdom