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JORDAN-MORPHISMS IN *-ALGEBRAS

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ABSTRACT. As a continuation of Størmer's work on Jordan-morphisms in C^*-algebras we consider Jordan-morphisms φ from *-algebras $\mathfrak{A}$ into the *-algebra $B(\mathcal{H})$, and assume that $\varphi(\mathfrak{A})$ is again a *-algebra. We then establish the existence of three mutually orthogonal central projections P_i, $i = 1,2,3$, in $\varphi(\ )''$ such that $P_1 + P_2 + P_3 = I$ and

$\varphi(\cdot)P_1$ is a morphism,

$\varphi(\cdot)P_2$ is an antimorphism.

P_3 is the largest projection such that $\varphi(\cdot)P_3$ is a morphism, as well as an antimorphism.

Uniqueness is also shown. The theorem improves a result of Kadison and Størmer. Our proofs are self-contained.

1. Introduction. Let $\mathfrak{A}$ be an associative *-algebra, and let * denote the *-operation, i.e., $a \to a^*$ is involutary, antilinear, and satisfies

$$(ab)^* = b^*a^*, \qquad a, b \in \mathfrak{A}.$$

Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ be the *-algebra of all bounded operators on $\mathcal{H}$. A linear mapping $\varphi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ between *-algebras is said to be a Jordan-morphism if

$$\varphi(a^*) = \varphi(a)^*$$

and

$$\varphi(\{a, b\}) = \{\varphi(a), \varphi(b)\}, \qquad a, b \in \mathfrak{A},$$

where $\{a, b\} = ab + ba$ is the anticommutator. Let $\varphi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ be linear satisfying $\varphi(a^*) = \varphi(a)^*$ for $a \in \mathfrak{A}$. We then say that φ is a morphism (resp., an antimorphism) if $\varphi(ab) = \varphi(a)\varphi(b)$ (resp., $\varphi(ab) = \varphi(b)\varphi(a)$) for $a, b \in \mathfrak{A}$.

In this note we prove the following result:

THEOREM. *Let $\mathfrak{A}$ be a *-algebra, and let $\varphi\colon \mathfrak{A} \to B(\mathcal{H})$ be a Jordan-morphism. Assume that $\varphi(\mathfrak{A})$ is again a *-algebra.*

Then there exist three mutually orthogonal projections P_i, $i = 1,2,3$, in the centre of the W^*-algebra generated by $\varphi(\mathfrak{A})$, such that:

(1) $\varphi(\cdot)P_1$ *is a morphism, and not an antimorphism*;

(2) $\varphi(\cdot)P_2$ *is an antimorphism, and not a morphism*;

(3) P_3 *is the largest projection such that $\varphi(\cdot)P_3$ is a morphism, as well as an antimorphism*;

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(4) $I = P_1 \oplus P_2 \oplus P_3$.

Conversely, the above conditions, (1)–(4), *determine the central projections* P_i (*uniquely*).

The decomposition problem for Jordan-morphisms was first considered by Jacobson and Rickart [1], Kadison [2, 3] and by Størmer [4]. Our theorem extends Størmer's result only at one point: the domain is allowed to be an arbitrary *-algebra instead of a C^*-algebra. But we feel that our technique is nonetheless of some interest.

The existing proofs of the decomposition theorem all go back to the key paper of Jacobson and Rickart, in which the domain of the morphism is required to be a matrix algebra; later results have proofs which reduce to this case. Our proof, instead, reduces to a special case for the range where the range (or its closure) is a prime ring, e.g. a factor. We are then able to use ideas of Herstein's [6] to complete the proof.

2. A lemma. The key to our proof is the following basic product formula:

$$[\varphi(ab) - \varphi(a)\varphi(b)][\varphi(cd) - \varphi(d)\varphi(c)] = 0, \qquad a, b, c, d \in \mathfrak{A}. \tag{5}$$

Of course, (5) follows easily from the theorem [4]. But the point here is that it is possible to derive (5) directly from the assumptions (rather than the conclusion) in the theorem. The following lemma is therefore central:

LEMMA. *Let $\mathfrak{A}$ be a *-algebra, and let φ: $\mathfrak{A} \to B(\mathcal{H})$ be a Jordan-morphism such that $\varphi(\mathfrak{A})$ is a *-algebra.*

Then $(\varphi(ab) - \varphi(a)\varphi(b))(\varphi(cd) - \varphi(d)\varphi(c)) = 0$ *for all quadruples of elements a, b, c, d in $\mathfrak{A}$.*

3. Two reductions in the proof of the Lemma.

REDUCTION 1. We may assume that the pair $(\mathfrak{A}, \varphi)$ is unital, i.e., $\mathfrak{A}$ has a unit $\mathbf{1}$, and $\varphi(\mathbf{1}) = I$.

If $(\mathfrak{A}, \varphi)$ is not unital, we may adjoin a unit in the usual manner: $\tilde{\mathfrak{A}} = \{(\lambda, a)$: $\lambda \in \mathbf{C}, a \in \mathfrak{A}\}$, and define $\tilde{\varphi}(\lambda, a) = \lambda I + \varphi(a)$.

REDUCTION 2. We need only consider the case where $\overline{\varphi(\mathfrak{A})}''$ is a factor.

In the general case we may consider the C^*-algebra $\overline{\varphi(\mathfrak{A})}$. Let P be the set of pure states of $\overline{\varphi(\mathfrak{A})}$, and let $\pi = \sum_{\omega \in P}^{\oplus} \pi_\omega$ be the corresponding atomic representation. We recall that $(\pi_\omega, \mathcal{H}_\omega)$ denotes the GNS-representation associated to the state ω, and it is known (easy) that π is faithful [5, Theorem 2.3.15]. For each $\omega \in P$, the map $\pi_\omega \circ \varphi$ is a Jordan-morphism into the factor $\pi_\omega(\varphi(\mathfrak{A}))'' = \pi_\omega(\overline{\varphi(\mathfrak{A})})''$.

4. Technical facts.

FACT 1. Let $\mathfrak{M}$ be a factor, and let x, y be a pair of elements in $\mathfrak{M}$. Assume that $x\mathfrak{M}y = \{0\}$. Then it follows that $x = 0$ or $y = 0$.

PROOF. If C_x (resp., C_y) denotes the respective central support, it follows immediately that $C_x C_y = 0$.

DEFINITION 4.1. Let $\mathfrak{A}$, $B(\mathcal{H})$, φ be as in the Theorem. For $a, b \in \mathfrak{A}$ we define

$$a^b = i[\varphi(ab) - \varphi(a)\varphi(b)] \quad \text{and} \quad a_b = i[\varphi(ab) - \varphi(b)\varphi(a)].$$

DEFINITION 4.2. For a *-algebra $\mathfrak{A}$ we set

$$\mathfrak{A}_{sa} = \{a \in \mathfrak{A} : a^* = a\}.$$

FACT 2. For any pair of elements a, b in $\mathfrak{A}_{sa}$ it follows that $a^b \in B(\mathcal{H})_{sa}$, and $a_b \in B(\mathcal{H})_{sa}$.

PROOF. Exploit the cancellations in the formulas for $(a^b)^* - a^b$, and $(a_b)^* - a_b$.

5. Proof of the Lemma. We assume that $\mathbf{1} \in \mathfrak{A}$, $\varphi(\mathbf{1}) = I$, and that $\varphi(\mathfrak{A})''$ is a factor. From [5, p. 208] we have the following identity:

$$a^b a_b = 0 \quad \text{for all } a, b \in \mathfrak{A}. \tag{6}$$

From [6] we take, directly, the product formula:

$$a^b\varphi(ab - ba)\varphi(c)\varphi(ab - ba)a_b = 0 \quad \text{for all } a, b, c \in \mathfrak{A}. \tag{7}$$

Formulas (6) and (7) may in fact be derived by pure algebra, using only that φ preserves the anticommutator.

Now, by (7), and Fact 1, we have that $a^b\varphi(ab - ba) = 0$ or $\varphi(ab - ba)a_b = 0$, whenever $a, b \in \mathfrak{A}$. Assume first that $a^b\varphi(ab - ba) = 0$. Then

$$\begin{aligned}\varphi(ab - ba) &= 2\varphi(ab) - \varphi(ab + ba)\\ &= 2\varphi(ab) - [\varphi(a)\varphi(b) + \varphi(b)\varphi(a)] = -i(a^b + a_b).\end{aligned}$$

Multiplying through by a^b, and using (6), we get $(a^b)^2 = a^b\varphi(ab - ba) = 0$.

Similarly, the second alternative yields the identity $(a_b)^2 = 0$. So, for all a, b in $\mathfrak{A}$, we have $(a^b)^2 = 0$ or $(a_b)^2 = 0$. (The argument up to this point is close to [6].)

We can now use Fact 2 to derive the next relation:

$$a^b = 0 \quad \text{or} \quad a_b = 0 \quad \text{for } a, b \in \mathfrak{A}_{sa}. \tag{8}$$

Hence,

$$a_b\varphi(c)a^b = 0 \quad \text{for} \quad a, b \in \mathfrak{A}_{sa} \text{ and } c \in \mathfrak{A}. \tag{9}$$

Polarizing (9), we see that $a_d\varphi(c)a^b + a_b\varphi(c)a^d = 0$, when a, b, $d \in \mathfrak{A}_{sa}$, and $c \in \mathfrak{A}$. Using first (8), and then Fact 1, we get

$$a_d = 0 \quad \text{or} \quad a^b = 0 \quad \text{for any triple of elements } a, b, d \in \mathfrak{A}_{sa}. \tag{10}$$

Polarizing $a_d a^b = 0$, we get

$$a_d c^b + c_d a^b = 0 \quad \text{for all } a, b, c, d \in \mathfrak{A}_{sa}. \tag{11}$$

As before, the alternatives in (10) imply that each term in (11) is zero, so that

$$a^b c_d = 0 \quad \text{for all } a, b, c, d \in \mathfrak{A}_{sa}. \tag{12}$$

From (12) the Lemma follows easily.

6. Proof of the Theorem. Define

$$\mathcal{Q}_1 = \{\varphi(ab) - \varphi(a)\varphi(b) : a, b \in \mathfrak{A}\}$$

and

$$\mathcal{Q}_2 = \{\varphi(ab) - \varphi(b)\varphi(a) : a, b \in \mathfrak{A}\}.$$

Let E_1, E_2 and E_3 denote the orthogonal projections onto $\bigcap_{A\in\mathfrak{A}_1}\ker A$, $\bigcap_{A\in\mathfrak{A}_2}\ker A$ and $\bigcap_{A\in\mathfrak{A}_1\cup\mathfrak{A}_2}\ker A$, respectively.

If P_A is the projection onto $\overline{\operatorname{Ran} A}$, we have

$$E_1 = \operatorname{glb}\{I - P_A : A \in \mathfrak{A}_1\}, \qquad E_2 = \operatorname{glb}\{I - P_A : A \in \mathfrak{A}_2\} \quad \text{and}$$
$$E_3 = \operatorname{glb}\{I - P_A : A \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}.$$

Hence, $I - E_1 = \operatorname{lub}\{P_A:\ A \in \mathfrak{A}_1\}$. Similarly, $I - E_2 = \operatorname{lub}\{P_A:\ A \in \mathfrak{A}_2\}$, and $I - E_3 = \operatorname{lub}\{P_A: A \in \mathfrak{A}_1 \cup \mathfrak{A}_2\}$.

Clearly, $E_i \in \varphi(\mathfrak{A})''$, $i = 1,2,3$. But the range spaces of E_i, $i = 1,2,3$, are univariant under the action of $\varphi(\mathfrak{A})$, so we also have $E_i \in \varphi(\mathfrak{A})'$, $i = 1,2,3$. Hence, the E_i's are central.

We have $E_3 \leqslant E_1$, and $E_3 \leqslant E_2$. In view of the Lemma, $(I - E_1)(I - E_2) = 0$. Moreover, $(I - E_1)A_2 = (I - E_2)A_1 = 0$ for all $A_1 \in \mathfrak{A}_1$, and $A_2 \in \mathfrak{A}_2$. Therefore, $(I - E_1) \oplus (I - E_2) \leqslant I - E_3$, and $[(I - E_1) \oplus (I - E_2)]A = A$ for all A in $\mathfrak{A}_1 \cup \mathfrak{A}_2$. Alternatively, $P_A \leqslant (I - E_1) \oplus (I - E_2)$ for all $A \in \mathfrak{A}_1 \cup \mathfrak{A}_2$.

As a consequence, $I - E_3 = (I - E_1) \oplus (I - E_2)$, and $P_1 = I - E_2$, $P_2 = I - E_1$, and $P_3 = E_3$. This concludes the proof of the existence part.

Uniqueness. Let X_i, $i = 1,2,3$, be central projections satisfying conditions (1)–(4) in the Theorem. Clearly, then $X_3 = P_3$, $X_2 \leqslant I - P_3$, and $X_1 \leqslant I - P_3$. Condition (4) yields $X_3 = P_3$, and $X_1 \oplus X_2 = I - P_3$.

We have $X_2 \leqslant E_2$, and $P_1 = I - E_2 \leqslant I - X_2$; and therefore $P_1 = P_1(I - P_3) \leqslant (I - X_2)(I - P_3) = X_1$.

In the same way we obtain $P_2 \leqslant X_2$. A second application of condition (4) yields $P_1 = X_1$ and $P_2 = X_2$.

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