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# FACTORISATION OF CHARACTERISTIC FUNCTIONS ON NONCOMMUTATIVE GROUPS

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ABSTRACT. A characteristic function, without idempotent factors, on a separable compact group is decomposed, modulo characters, as a product of indecomposable characteristic functions and an infinitely divisible characteristic function.

A continuous normalized positive definite function on a topological group $G$ will be called a characteristic function. Denote by $|\phi|^2$ the characteristic function defined by $|\phi|^2(g) = |\phi(g)|^2$ for all $g$ in $G$. The characteristic function identically 1 will be called degenerate. A continuous homomorphism of $G$ to $\mathbf{C}^*$, the group of complex numbers modulo 1, will be called a character. We are concerned with the factorisation of a characteristic function as a product of characteristic functions where we write $\phi = \phi_1\phi_2$ if $\phi(g) = \phi_1(g)\phi_2(g)$ for all $g$ in $G$. A characteristic function $\phi$ is called indecomposable if it cannot be expressed as a product of two other characteristic functions, idempotent if $\phi = \phi^2$ and infinitely divisible if for each $n \in \mathbf{N}$ one may write $\phi = \prod_{i=1}^n \phi_i^{(n)}$ for some characteristic function $\phi^{(n)}$, each $\phi_i^{(n)} = \phi^{(n)}$. Denote the set of factors of $\phi$ by $F_\phi$, the set of indecomposable factors of $\phi$ by $IF_\phi$ and the subgroup of $G$ generated by $\{g: \phi(g) \neq 0\}$ by $G_\phi$. Denote left Haar measure on a separable locally compact group by $dg$.

For the purposes of factorisation we shall consider two characteristic functions $\phi_1$ and $\phi_2$ to be equivalent if $\phi_1 = \phi_2\chi$ where $\chi$ is a character. When $G$ is commutative a characteristic function is the Fourier transform of a probability measure on the dual group $\hat{G}$ and equivalent characteristic functions are the Fourier transforms of shift-equivalent measures on $\hat{G}$ [**4**].

A. I. Khinchin [**2**] showed that the characteristic function of a probability distribution on $\mathbf{R}$ can be represented as $\phi_2\phi_3$ where $\phi_2$ is a denumerable product of indecomposable factors, $\phi_3$ has no indecomposable factors and is necessarily infinitely divisible. K. R. Parthasarathy, R. Ranga Rao and S. R. S. Varadhan [**3**] extended this result to a characteristic function on an arbitrary separable locally compact commutative group decomposing it as $\phi_1\phi_2\phi_3$ where $\phi_1$ is idempotent, $\phi_2$ and $\phi_3$ as above. When the group has no compact subgroups there is no proper idempotent factor.

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The factorisation can be translated to the positive-definite matrices $[\alpha_{ij}] = [\phi(g_i g_j^{-1})]$ for sequences $(g_i)$ in $G$. The product of characteristic functions corresponds to coefficientwise multiplication of the matrices, and matrices $[\alpha_{ij}]$ and $[\beta_{ij}]$ correspond to equivalent characteristic functions if and only if $\alpha_{ij} = \beta_{ij} c_i c_j$ for $c_i, c_j \in \mathbf{C}^*$.

In §1 we consider the cancellation of idempotent factors from a characteristic function on a topological group and find conditions determining whether a characteristic function has idempotent factors or not. In §2 we prove Khinchin's factorisation theorem for a characteristic function, without idempotent factors, on a separable compact group. We have not been able to prove Khinchin's theorem for characteristic functions with idempotent factors, neither have we been able to construct a counterexample. In §3 we show why, in the commutative case, any characteristic function can be factorised as above.

## 1. Idempotent factors of a characteristic function on a topological group.

PROPOSITION 1. *Let $G$ be a topological group. If $\psi$ is an idempotent factor of a characteristic function $\phi$ of $G$ then $\psi = \chi_H$ where $H$ is an open and closed subgroup of $G$. The maximal idempotent factor, i.e. that with the minimal support and so the least degenerate, is $\chi_{G_\phi}$. One factorises $\phi$ as $\chi_{G_\phi}\phi_0$ where $\phi_0$ is the restriction of $\phi$ to $G_\phi$.*

PROOF. An idempotent is necessarily of the form $\chi_H$ for a subset $H$ of $G$. Since the factors are required to be continuous it follows that $H$ is open and closed and, since $\psi(g_1) = 1$, $\psi(g_2) = 1$ implies $\psi(g_1 g_2) = 1$, [**1**, 32.7], it follows that $H$ must be an open (and closed) subgroup. For $\chi_H$ to be a factor of $\phi$, necessarily $\chi_H(g) \neq 0$ whenever $\phi(g) \neq 0$, so $H \supset G_\phi$. By construction $G_\phi$ is an open subgroup and so also closed. By [**1**, 32.43], $\phi_0$ is also a characteristic function for $G$.

COROLLARY 1. *The characteristic function $\phi$ has nondegenerate idempotent factors if and only if $G_\phi \neq G$.*

PROPOSITION 2. *Let $\phi$ be an infinitely divisible characteristic function on a group $G$. It has a nondegenerate idempotent factor if and only if it has zeros.*

PROOF. The function $\phi_1 = \lim_n |\phi|^{2^{-n}}$ is the idempotent factor, where $|\phi|(g) = |\phi(g)|$ for all $g$ in $G$. Indeed, $\phi\bar{\phi}$ is a characteristic function and, since $\phi$ is infinitely divisible, its repeated square roots will exist and be characteristic functions, Furthermore $\phi_1 = \chi_{G_\phi}$.

This generalises Lemma 4.2 of [**5**], proved there for compact $G$.

## 2. Characteristic functions, without idempotent factors, on a separable compact group.

LEMMA 1. *Let $G$ be a separable compact group and $(\phi_n)$ a sequence of characteristic functions such that $\int_G |\phi_n(g)|^2\, dg \to 1$ as $n \to \infty$. Then there exists a sequence $(\chi_n)$ of characters such that $\phi_n \chi_n$ converges uniformly to the degenerate characteristic function as $n \to \infty$.*

PROOF. The proof is contained explicitly in the proof of [5, Lemma 4.1].

For a characteristic function $\phi$, without idempotent factors, on a compact group we define the Khinchin functional $N_\phi$ on $F_\phi$, which measures 'departure' from the degenerate characteristic functional, by $N_\phi(\psi) = -\int_G \log|\psi(g)|\,dg$. It is well defined and convergent since $G$ is generated by a sequence of elements $(g_i)$ such that $\phi(g_i) \neq 0$ for all $i$, and since $G$ is compact, $N_\phi$ is bounded.

PROPOSITION 3. *Let $\phi$ be a characteristic functional without idempotent factors on a separable compact group $G$. If $\psi_1, \psi_2, \psi \in F_\phi$*

(i) $N_\phi(\psi_1\psi_2) = N_\phi(\psi_1) + N_\phi(\psi_2)$,

(ii) $N_\phi(\psi) \geq \int_G(1 - |\psi(g)|)\,dg \geq 0$,

(iii) $N_\phi(\psi) = 0$ *if and only if $\psi$ is equivalent to the degenerate characteristic function.*

PROOF. Properties (i) and (ii) are obvious. Property (iii) follows since $N_\phi(\chi_G) = 0$ and $N_\phi(\chi) = 0$ for any character $\chi$; if $N_\phi(\psi) = 0$ then

$$\int_G (1 - |\psi(g)|^2)\,dg \leq 2\int_G (1 - |\psi(g)|)\,dg = 0$$

by (ii), and, by Lemma 1, there exists a character $\chi$ such that $\psi\chi$ is degenerate.

LEMMA 2. *Let $\phi$ be a characteristic function without idempotent factors on a separable compact group $G$ and let $(\psi_i)$ be a sequence of factors of $\phi$ such that for all $n \in \mathbf{N}$ the product $\prod_{i=1}^n \psi_i$ is also a factor of $\phi$. Then there exist characters $\chi_i$ such that $\prod_{i=1}^n \psi_i\chi_i$ converges to a characteristic function as $n \to \infty$.*

PROOF. $\sum N_\phi(\psi_i) \leq N_\phi(\phi)$ so $\sum_{i=k}^\infty N_\phi(\psi_i)$ and $N_\phi(\prod_{i=k}^\infty \psi_i)$ converge to zero as $k \to \infty$. By Lemma 1 there exist $(\chi_k)$ such that $(\chi_{k-1}\prod_{i=k}^\infty \psi_i)$ converges to the degenerate characteristic function as $k \to \infty$. Thus, absorbing each $\chi_{k-1}$ in the preceding finite product, $\prod_{i=1}^n \psi_i\chi_i$ converges to a characteristic function as $n \to \infty$.

PROPOSITION 4. *Let $\phi$ be a real-valued characteristic function on a compact group. Every sequence in $F_\phi$ has a convergent subsequence.*

PROOF. The set $F_\phi$ is equicontinuous since, for $\psi \in F_\phi$,

$$|\psi(g) - \psi(h)|^2 \leq 2(1 - \operatorname{Re}\psi(g^{-1}h)) \leq 2(1 - \phi(g^{-1}h)).$$

The proposition follows from the Arzela-Ascoli theorem.

Lemma 2 is the noncommutative version of [4, Theorem III.5.3], Proposition 4 is an analogue of Corollary III.5.2.

PROPOSITION 5. *Let $G$ be a separable compact group. Any characteristic function without idempotent factors can be factorized, modulo a character, as a product of a denumerable number of indecomposable characteristic functions and a characteristic function with no indecomposable factors.*

PROOF. If $\phi$ does not have any indecomposable factors the proposition holds. Suppose $\phi$ has indecomposable factors. Write $\operatorname{Sup}\{N_\phi(\psi)\colon \psi \in IF_\phi\} = \delta(\phi)$. One can decompose $\phi$ as $\psi_1\lambda_1$ where $N_\phi(\psi_1) \geq \frac{1}{2}\delta(\phi)$ and decompose the characteristic function $\lambda_{n-1}$ as $\psi_n\lambda_n$ where $N_\phi(\psi_n) \geq \frac{1}{2}\delta(\lambda_{n-1})$, for $n = 2, 3, \ldots$. If $\lambda_k$ has no indecomposable factors for some $k$ the process terminates and the proposition holds. When the process does not terminate there exist, by Lemma 2, characters $\chi_i$ such that $\prod \psi_i \chi_i$ converges. So $N_\phi(\psi_n) \to 0$ as $n \to \infty$. So also $\lambda_n$ will converge to a characteristic function $\lambda$ as $n \to \infty$. If $\lambda$ has an indecomposable factor $\psi$ then $\psi \in F_{\lambda_n}$ for all $n$ and so $N_\phi(\psi) \leq \delta(\lambda_n)$ for all $n$; as $\delta(\lambda_n) \leq 2N_\phi(\psi_{n+1}) \to 0$ as $n \to \infty$, it follows from Proposition 3 that $\psi$ is a character.

LEMMA 3. *Let $\phi$ be a characteristic function, with no indecomposable factors and with no idempotent factors, on a separable compact group $G$. There exists a sequence of decompositions $(D_n)$ of $\psi$ such that $\nu = \inf_n \sup\{1 - |\psi(g)| : \psi \in D_n, g \in G\} = 0$.*

PROOF. For any decomposition $D$ of $\phi$ let

$$\nu_D = \sup\{1 - |\psi(g)| : \psi \in D, g \in G\}.$$

For any characteristic function $\tau$, if $\psi \in F_\tau$ then $1 - |\psi(g)| \leq 1 - |\tau(g)|$ for all $g$ in $G$. One can arrange an array of decompositions

$$\left(D_n : \phi = \phi_{n,1} \cdots \phi_{n,k_n}\right)$$

such that $\nu_{D_n} \to \nu$ as $n \to \infty$, $1 - |\phi_{n,j}(g)| \leq 1 - |\phi_{n,1}(g)|$ for all $g \in G$, $1 < j \leq k_n$, and $1 - |\phi_{n,1}(g)| = \nu_{D_n}$ for some $g$. Using Lemma 2, $\phi$ can be decomposed as $\phi_1\phi_2$, where $\phi_2 = \lim_n \prod_{j=2}^{k_n} \chi_{n,j}\phi_{n,j}$, for an array $(\chi_{n,i})$ of characters of $G$, and such that $1 - |\phi_1(g)| = \nu$ for some $g$. Since $\phi_1$ and $\phi_2$ are again decomposable $\nu$ must be 0.

An array of decompositions $(D_n)$ such that $\nu = 0$ will be called uniformly infinitesimal.

COROLLARY 2. *If a characteristic function $\phi$ on a compact separable group has neither idempotent nor indecomposable factors then $\{g : \phi(g) \neq 0\} = G_\phi$.*

PROOF. Since $G_\phi = G_{|\phi|^2}$ it is sufficient to prove that if $|\phi|^2(g_1) > 0$ and $|\phi|^2(g_2) > 0$ then $|\phi|^2(g_1 g_2) > 0$. Choose a uniformly infinitesimal array of decompositions $(\phi_{n,1} \cdots \phi_{n,k_n})_n$ of $\phi$. For each of the decompositions $|\phi|^2(g) = |\phi_{n,1}|^2(g) \cdots |\phi_{n,k_n}|^2(g)$. Thus $|\phi|^2(g) > 0$ if and only if, for any $n$, $|\phi_{n,j}|^2(g) > 0$, $1 \leq j \leq k_n$, and so also if and only if $\lim_n(n - |\phi_{n,j}|^2(g)) < \infty$ for $j \in \mathbf{N}$. The corollary follows using [**5**, Lemma 3.6].

PROPOSITION 6. *A characteristic function $\phi$, with neither idempotent nor indecomposable factors, on a compact separable group $G$, is, modulo a character, infinitely divisible.*

PROOF. By Corollary 1, $G = G_\phi$. We denote $\phi(h^{-1}g)(\phi(h^{-1})\phi(g))^{-1}$ by $K(g, h)$, adding suffixes if required. By Lemma 3 we can find a uniformly infinitesimal array $(\phi_{n,1} \cdots \phi_{n,k_n})_n$ of decompositions of $\phi$ such that, for large enough $n$, $1 - |\phi_{n,j}(g)|$ is as small as we like. By [**6**, Lemma 3.5],

$$|K_{n,j}(g, h)| \leq 2\left(1 - |\phi_{n,j}(h^{-1})|\right)^{1/2}\left(1 - |\phi_{n,j}(g)|\right)^{1/2}\left(\phi_{n,j}(h^{-1})\phi_{n,j}(g)\right)^{-1}$$

for $n \in \mathbf{N}$, $1 \leq j \leq k_n$. So $\lim_n \sup_j |1 - K_{n,j}(g, h)| = 0$. Using the procedure of [**6**, Lemma 4.2] we can define $L(g, h) = \operatorname{Log} K(g, h)$ and prove it to be continuous and positive-definite on $G \times G$. As in [**6**, Lemma 4.3], $L(h, g^{-1})$ is an additive 2-cocycle.

It is a coboundary since $H_2(G, \mathbf{R}) = \{0\}$ and the real and imaginary parts of $L$ can be considered separately. Hence $L(g, h) = \psi(h^{-1}g) - \psi(h^{-1}) - \psi(g)$ for some continuous conditionally positive-definite function $\psi$ on $G$. By [**5**, Theorem 4.1], $e^{\psi}$ is infinitely divisible. As in the proof of [**6**, Theorem 5.1], $e^{\psi} = \phi\chi$ for some character $\chi$ of $G$.

COROLLARY 3. *On a separable compact group, if a characteristic function has no idempotent factors then it has indecomposable factors whenever it has zeros.*

PROOF. Suppose $\phi$ has zeros but no indecomposable factors. By Proposition 6 it is infinitely divisible so by Proposition 2 it cannot have zeros.

THEOREM. *Let $G$ be a separable compact group and $\phi$ a characteristic function on $G$ with no idempotent factors. Then $\phi$ can be decomposed, modulo a character, as a product of indecomposable characteristic functions and an infinitely divisible characteristic function.*

PROOF. The theorem follows from Propositions 5 and 6.

**3. Commutative groups.** The method in [**3**] for proving Lemma 2 for a locally compact separable commutative group is to use [**4**, Corollary III.5.2], the analogue of our Proposition 4, to prove the existence of characters $\chi_i$ such that products $\prod_i \psi_i\chi_i$ converge, and [**4**, Theorem III.5.2] to prove that all such convergent products are equivalent. Lemma 5 is [**4**, Theorem III.5.2] with a simpler proof than the original.

LEMMA 4. *Let $G$ be a complete separable metric commutative group. If $\phi$ and $\psi$ are characteristic functions on $G$ such that $\phi\psi$ is the degenerate characteristic function, then $\phi$ and $\psi$ are characters.*

PROOF. Denote the measure corresponding to a characteristic function $\zeta$ by $\mu_\zeta$. Since $\mu_\phi * \mu_\psi$ is the unit mass at the neutral element of $G$, so $\mu_\phi$ and $\mu_\psi$ must be point masses. Hence $\phi$ and $\psi$ are characters.

LEMMA 5. *Let $\phi$ and $\psi$ be characteristic functions on a complete separable metric commutative group. If $\phi \in F_\psi$ and $\psi \in F_\phi$, then $\phi$ is equivalent to $\psi$.*

PROOF. The lemma follows from Lemma 4. Indeed, if $\phi = \chi_1\psi$ and $\psi = \chi_2\phi$, then $\phi = \chi_1\chi_2\phi$, so $\chi_1\chi_2$ is degenerate.

LEMMA 6. *Let $\phi$ be a characteristic function on a locally compact separable commutative group $G$. Any character on $G_\phi$ extends uniquely to a character of $G$.*

PROOF. Denote the annihilator of $G_\phi$ in $\hat{G}$ by $K$ and identify $\hat{G}_\phi = \hat{G}/K$ with a Borel section $B$ of $G$. As an element of $B$ is also an element of $G$, a character of $G_\phi$ uniquely determines a character of $G$.

Let $\phi$ be a characteristic function on a locally compact separable commutative group $G$. By Proposition 1, $\phi$ can be factorised as $\chi_{G_\phi}\phi_0$. Propositions 5 and 6 hold for $\phi_0$ on $G_\phi$ [**3**]. By Lemma 6 the characters of $G_\phi$ occurring in the factorisation extend to characters of $G$. Thus $\phi$ can be factorised as $\phi_1\phi_2\phi_3$ as stated in the introduction.

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