PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

A REMARK ON LINEAR DIFFERENTIAL SYSTEMS WITH THE SAME INVARIANT SUBBUNDLES

ROBERT E. VINOGRAD[1]

ABSTRACT. Given a flow $\sigma(\omega,t)=\omega\cdot t$ on a compact metric space Ω and a continuous $n\times n$-matrix $A(\omega)$, the family of ODE systems $\dot{x}=A(\omega\cdot t)x$ defines a linear skew-product flow on $W=\Omega\times X$, $X=R^n$ or C^n. Let $W=U\oplus V$ be a Whitney sum and P: $W\to W$ be the correspondent projector. Result: the subbundles U and V are invariant for the flows induced by $A(\omega)$ and $B(\omega)$ iff $A(\omega)-B(\omega)$ commutes with $P(\omega)$ for all $\omega\in\Omega$.

As is known (e.g. see [**1**]), given a flow $\sigma(\omega,t)=\omega\cdot t$ on a space Ω (usually compact metric) and an $n\times n$-matrix $A(\omega)$ which is continuous on Ω, the family of systems

$$\dot{x}=A(\omega\cdot t)x, \qquad \cdot=\frac{d}{dt}, x\in X=R^n \text{ or } C^n, \tag{1}$$

naturally defines a flow (so-called LSPF [**1**]) on the product space $W=\Omega\times X$. Let $W=U\oplus V$ be a Whitney sum subbundle decomposition and X_ω, U_ω, V_ω be corresponding fibers so that $X_\omega=U_\omega\oplus V_\omega$. Then there is an uniquely determined projector P: $W\to W$ with range $P=U$ and null $P=V$ (i.e. $P(\omega)=P|X_\omega$ is a continuous projection $X_\omega\to X_\omega$ with range $P(\omega)=U_\omega$, null $P(\omega)=V_\omega$, so that $P(\omega)u=u$, $P(\omega)v=0$ for $u\in U_\omega, v\in V_\omega$).

The subbundles U,V are said to be invariant for (1) if every solution of (1) starting in U or V remains in it for all t. If this is the case, we shall say for brevity that P is invariant for (1).

The purpose of the present paper is to establish the following surprisingly simple result.

THEOREM. *P is invariant simultaneously for two systems* (1) *and*

$$\dot{y}=B(\omega\cdot t)y \tag{2}$$

if and only if $B(\omega)-A(\omega)$ commutes with $P(\omega)$ for all $\omega\in\Omega$.

LEMMA. (i) *P is invariant for* (1) *iff $\tilde{x}(t)=P(\omega\cdot t)x(t)$ is a solution to* (1) *whenever $x(t)$ is.* (ii) *If P is invariant for* (1), *then $\dot{P}(\omega\cdot t)$ exists and* (*suppressing $\omega\cdot t$*)

$$\dot{P}=AP-PA \quad \text{for all } \omega \text{ and } t. \tag{3}$$

(iii) *If $P(\omega)$ is a continuous projection on Ω such that $\dot{P}(\omega\cdot t)$ exists and satisfies* (3), *then P is invariant for* (1).

Received by the editors November 30, 1981.

1980 *Mathematics Subject Classification*. Primary 34C35, 54H20, 58F10; Secondary 34K20.

Key words and phrases. Almost periodic, ordinary differential equations, projector, skew-product flow, subbundle, Whitney sum.

[1]This research was supported by NSF grant MCS 8115323.

PROOF. In what follows we denote by $\xi(t)$ a solution with $\xi(0) = \xi$.

(i) $\rightarrow$ Pick $u \in U_\omega$, $v \in V_\omega$ and consider the solutions $u(t)$, $v(t)$. By assumption, $\tilde{u}(t) = P(\omega \cdot t)u(t)$, $\tilde{v}(t) = P(\omega \cdot t)v(t)$ are also solutions, and we have $\tilde{u}(0) = P(\omega)u = u$, $\tilde{v}(0) = P(\omega)v = 0$. By uniqueness, $\tilde{u}(t) \equiv u(t)$, $\tilde{v}(t) \equiv 0$ which means $u(t) \in U_{\omega \cdot t}$, $v(t) \in V_{\omega \cdot t}$.

$\leftarrow$ Pick any solution $x(t)$, $x(0) = x \in X_\omega$; then $x = u + v$, $u \in U_\omega$, $v \in V_\omega$. Since U, V are invariant, we have $u(t) \in U_{\omega \cdot t}$, $v(t) \in V_{\omega \cdot t}$. Also $x(t) = u(t) + v(t)$, and so $P(\omega \cdot t)x(t) = u(t)$, a solution to (1).

(ii) Fix an arbitrary $\omega \in \Omega$ and a basis $x_1, \dots, x_n$ in X_ω. Let $X(t)$ be a fundamental matrix of (1) with vector-columns $x_1(t), \dots, x_n(t)$. By (i), the invariancy of P implies that $\tilde{X}(t) = P(\omega \cdot t)X(t)$ is again a fundamental matrix of (1). Hence (suppressing t and $\omega \cdot t$) $P = \tilde{X}X^{-1}$ is differentiable and

$$\dot{P} = \dot{\tilde{X}}X^{-1} + \tilde{X}(X^{-1})^{\cdot} = A\tilde{X}X^{-1} - \tilde{X}X^{-1}A = AP - PA.$$

(iii) Let (3) be given and $x(t)$ satisfy (1). Then

$$(Px)^{\cdot} = \dot{P}x + P\dot{x} = (AP - PA)x + PAx = APx,$$

i.e. $\tilde{x} = Px$ is a solution to (1). Now (i) implies (iii).

PROOF OF THEOREM. Let $(B - A)P = P(B - A)$ and P be invariant for (1). Then by (ii)

$$\dot{P} = AP - PA = AP - PA + [(B - A)P - P(B - A)] = BP - PB,$$

and hence by (iii) P is invariant for (2). Conversely, if P is invariant for both (1) and (2), then by (ii) $AP - PA = \dot{P} = BP - PB$ and so $(B - A)P = P(B - A)$.

REMARKS. (a) An obvious generalization for a Whitney sum with $m > 2$ addends, i.e., $W = V_1 \oplus \cdots \oplus V_m$ with projectors P_i where range $P_i = V_i$, null $P_i = \bigoplus_{k \neq i} V_k$, says: $V_1, \dots, V_m$ are simultaneously invariant for both (1) and (2) iff $B(\omega) - A(\omega)$ commutes with every $P_i(\omega)$, $i = 1, \dots, m$, for all $\omega \in \Omega$. (b) The last statement does not prevent one or both of the systems (1) and (2) from also having "finest" decompositions $V_i = V_i^{(1)} \oplus \cdots \oplus V_i^{(m_i)}$ which need not be common to (1) and (2).

REFERENCES

1. R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTH DAKOTA STATE UNIVERSITY, FARGO, NORTH DAKOTA 58105