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SOME CHARACTERIZATIONS OF WEAK RADON-NIKODYM SETS

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ABSTRACT. Let K be a weak*-compact convex subset of the dual E^* of a Banach space E. It is shown that K has the weak Radon-Nikodym property if and only if every x^{**} in E^{**} restricted to K is universally measurable if and only if every x^{**} in E^{**} restricted to any weak*-compact subset M of K has a point of continuity on (M, weak^*) if and only if K is a set of complete continuity if and only if every subset of K is weak* dentable in $(M, \sigma(E^*, E^{**}))$.

A subset of K of a Banach space E is said to have the weak Radon-Nikodym property (resp., a set of complete continuity) if for every finite measure space (Ω, Σ, μ) and bounded operator $T\colon L^1(\Omega, \Sigma, \mu) \to E$ satisfying $T(1_A/\mu(A)) \in K$ for every $A \in \Sigma$ with $\mu(A) \neq 0$ is represented by a Pettis kernel with values in K (resp. is a Dunford-Pettis operator).

If Ω is a compact Hausdorff space, then a real valued function defined on Ω is universally measurable if ϕ is μ-measurable for every Radon probability measure on Ω.

The set of all Radon probability measures on Ω will be denoted by $M_+^1(\Omega)$.

If K is a weak*-compact convex subset of the dual E^* of a Banach space E, we say that a universally measurable affine function f on $(K, \sigma(E^*, E))$ satisfies the barycentric formula if for any λ in $M_+^1(K, \sigma(E^*, E))$ we have

$$f\left(w^* - \int_K x^* \, d\lambda\right) = \int_K f(x^*) \, d\lambda,$$

where $w^*\text{-}\int_K x^* \, d\lambda$ is the barycenter of λ.

In this paper we are going to give a characterization of weak*-compact convex sets K that have the weak Radon-Nikodym property. Namely, we will show that such a set has the weak Radon-Nikodym property if and only if K is a set of complete continuity if and only if every x^{**} in E^{**} is universally measurable on (K, weak^*) if and only if every x^{**} in E^{**} restricted to any weak*-compact subset M in K has a point of continuity.

In the case where $K = B_{E^*}$, the unit ball of E^*, the above characterizations were given by Pelczynski **[9]**, Hagler **[5]**, Haydon **[6]**, Janicka **[7]**, Odell and Rosenthal **[8]** and Saab and Saab **[12** and **13]**.

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In the case where K is any absolutely convex weak*-compact subset, the above characterizations were given by Riddle, E. Saab and Uhl [**10**].

Our results yield a positive solution of a problem raised in [**10**, Problem 1]. Namely, every $x^{**} \in E^{**}$ restricted to every weak*-compact subset of K has a point of continuity on (M, weak*) whenever K has the weak Radon-Nikodym property.

Another consequence is that a weak*-compact subset M of a weak Radon-Nikodym set K is a weak Radon-Nikodym set, and if S is any linear operator from any Banach space F to E then $S^*(K)$ is a weak Radon-Nikodym set. We will also show that a weak*-compact convex set has the weak Radon-Nikodym property if and only if every bounded set in K is weak*-dentable in $(E^*, \sigma(E^*, E^{**}))$.

We would like to thank N. Kalton for showing us how to split the operator T in the proof of the next theorem, using a Lindenstrauss compactness argument.

THEOREM 1. *Let K be a w^*-compact convex subset of the dual E^* of a Banach space E. The following statements are equivalent*:

(i) *Every bounded sequence $(x_n)_{n \geq 1}$ in E has a subsequence $(x_{n_p})_{p \geq 1}$ such that for every $x^* \in K$, $\lim_p x^*(x_{n_p})$ exists.*

(ii) *The restriction of every $x^{**} \in E^{**}$ to $(K, \sigma(E^*, E))$ is universally measurable.*

(iii) *The restriction of every $x^{**} \in E^{**}$ to $(K, \sigma(E^*, E))$ is universally measurable and satisfies the barycentric formula.*

(iv) *For every w^*-compact subset M in K, the restriction of every $x^{**} \in E^{**}$ to $(M, \sigma(E^*, E))$ has a point of continuity.*

(v) *The set K has the weak Radon-Nikodym property.*

PROOF. (i) $\Rightarrow$ (ii). Let x^{**} in E^{**}, choose $(x_\alpha)_{\alpha \in I}$ a net E such that $\|x_\alpha\| \leq \|x^{**}\|$ and $(x_\alpha)_{\alpha \in I}$ converging weak* to x^{**}. Let $A = \{x_\alpha \,|\, K;\ \alpha \in I\} \subseteq C(K)$, where $C(K)$ is the space of continuous functions on $(K, \sigma(E^*, E))$. (i) and Theorem 2F of [**3**] imply the existence of a universally measurable function g on $(K, \sigma(E^*, E))$ and a subnet (x_β) of (x_α) such that $g(x^*) = \lim_\beta x^*(x_\beta)$ for every x^* in K. But this implies that x^{**} restricted to K is equal to g. Hence, x^{**} restricted to $(K, \sigma(E^*, E))$ is universally measurable.

(ii) $\Rightarrow$ (i). Fix $\lambda \in M_+^1(K)$, and let $(x_n)_{n \geq 1}$ be a bounded sequence in E. Consider $(x_n)_{n \geq 1}$ as continuous functions on K. Every subsequence $(x_{n_p})_{p \geq 1}$ of $(x_n)_{n \geq 1}$ has a λ-measurable cluster point namely any x^{**} restricted to K that is a cluster point of (x_{n_p}) in $(E^{**}, \sigma(E^{**}, E^*))$. Apply Theorem 2F of [**3**] to deduce that (x_n) must have a subsequence (x_{n_p}) such that $\lim_p x^*(x_{n_p})$ exists for every x^* in K.

(i) $\Rightarrow$ (iv). Let x^{**} in E^{**} and choose a net $(x_\alpha)_{\alpha \in I}$ in E such that $x^{**} = w^*\text{-}\lim_\alpha x_\alpha$ and $\|x_\alpha\| \leq \|x^{**}\|$. Let $A = \{x_\alpha |_K;\ \alpha \in I\}$. If there exists a w^*-compact subset M of K such that x^{**} restricted to $(M, \sigma(E^*, E))$ does not have any point of continuity, then by [**8**] there exists a sequence $(x_n)_{n \geq 1}$ in E such $f_n = x_n |_K$ belongs to A and $(f_n)_{n \geq 1}$ is equivalent to the usual l_1-basis in $C(K)$, but this contradicts (i).

(iv) $\Rightarrow$ (i). Fix $(x_n)_{n \geq 1}$ a bounded sequence in E. (iv) and [**10**] imply that any x^{**} restricted to any w^*-compact subset M of $K - K$ has a point of continuity on $(M, \sigma(E^*, E))$. Without loss of generality we can assume that $0 \in K$. Put $H = K - K$ and let $T\colon E \to C(H)$ be defined by $Tx = x|_H$. It is easy to see that $T^*(B_{E^*}) = H$.

Hence T factors through a Banach space not containing l_1 [12] and therefore $\{T(x_n); n \geq 1\}$ is weakly precompact by [10]. This means that there is a subsequence (x_{n_p}) such that $Tx_{n_p} = x_{n_p|K}$ converges pointwise on K, that is $\lim_p x^*(x_{n_p})$ exists for every $x^* \in K$.

(i) $\Rightarrow$ (iii). Let $x^{**} \in E^{**}$, x^{**} is universally measurable as a function on K and its restriction to any subcompact M of K has a point of continuity so x^{**} satisfies the barycentric formula by a theorem of Choquet (see [1]).

(iii) $\Rightarrow$ (v) is [7].

To complete the proof we need to show that (v) $\Rightarrow$ (iv). By [10] it is enough to show that $K - K$ has the weak Radon-Nikodym property. Let $T\colon L^1[0,1] \to E^*$ such that $T(1_A/\lambda(A)) \in K - K$ for every measurable set A such that $\lambda(A) > 0$. For every $n \geq 0$, let A_n be the algebra generated by the nth diadic partition of $[0,1]$, $\{I_n^1, I_n^2, \ldots, I_n^{2^n}\}$. For every $1 \leq j \leq 2^n$ we have that

$$T\left(\frac{1_{I_n^j}}{\lambda(I_n^j)}\right) = U_n\left(\frac{1_{I_n^j}}{\lambda(I_n^j)}\right) - V_n\left(\frac{1_{I_n^j}}{\lambda(I_n^j)}\right).$$

(Pick any choice.)

This enables us to define two linear operators

$$U_n\colon L_1(A_n) \to X^*$$

and

$$V_n\colon L_1(A_n) \to X^*$$

such that $T = U_n - V_n$ on $L_1(A_n)$, $U_n(1_A/\lambda(A)) \in K$ and $V_n(1_A/\lambda(A)) \in K$ for every $A \in A_n$ and $\lambda(A) \neq 0$. Let $\alpha = \sup_{x^* \in K} \|x^*\|$, then $\|U_n\| \leq \alpha$ and $\|V_n\| \leq \alpha$. Let $H = \bigcup_{n=0}^{\infty} L_1(A_n)$. By a Lindenstrauss compactness argument one can find two bounded linear operators $\tilde{U}$, $\tilde{V}\colon H \to X^*$ such that $T = \tilde{U} - \tilde{V}$ on H, $\tilde{U}(1_A/\lambda(A)) \in K$ and $\tilde{V}(1_A/\lambda(A)) \in K$ for any $A \in \bigcup_{n=1}^{\infty} A_n$ and $\lambda(A) \neq 0$. Let U and V denote the unique extensions of $\tilde{U}$ and $\tilde{V}$ respectively to $L^1[0,1]$. It is clear that $T = U - V$, $U(1_A/\lambda(A)) \in K$ and $V(1_A/\lambda(A)) \in K$ for every measurable set A such that $\lambda(A) \neq 0$. Let g_1 and g_2 be two Pettis integrable functions,

$$g_1\colon [0,1] \to K, \qquad g_2\colon [0,1] \to K,$$

such that

$$U(f) = \text{Pettis-}\int_0^1 f g_1 \, d\lambda$$

and

$$V(f) = \text{Pettis-}\int_0^1 f g_2 \, d\lambda$$

for every $f \in L^1[0,1]$. Hence

$$T(f) = \text{Pettis-}\int_0^1 f(g_1 - g_2)\, d\lambda \quad \text{for every } f \in L^1[0,1].$$

This shows that $K - K$ has the weak Radon-Nikodym property for the unit interval. Therefore $K - K$ is a set of complete continuity by **[10]** and hence, $K - K$ has the weak Radon-Nikodym property **[10]**.

A set K that satisfies (iv) is said to have the scalar point of continuity.

COROLLARY 2. *Let K be a w^*-compact convex subset of the dual E^* of a Banach space E. If K has the WRNP then*:

(i) *any w^*-compact convex subset M of K has the weak Radon-Nikodym property*;

(ii) *for any bounded linear operator T from a Banach space F to E, $T^*(K)$ has the weak Radon-Nikodym property in F^*.*

PROOF. (i) For every x^{**} in E^{**}, x^{**} restricted to M is universally measurable.

(ii) Let $y^{**} \in F^{**}$ and $\lambda \in M_+^1(T^*(K))$. Choose $\mu \in M_+^1(K)$ such that $\lambda = T^*(\mu)$ **[2]**, the linear functional $y^{**}T^* \in E^{**}$. Hence, its restriction to K is μ-measurable. Therefore $y^{**}_{|T^*(K)}$ is $\lambda = T^*(\mu)$ measurable **[2]**. Consequently, y^{**} is universally measurable as a function on $T^*(K)$.

COROLLARY 3. *For any weak*-compact convex subset K of the dual E^* of a Banach space E, the following statements are equivalent*:

(i) *The set K has the weak Radon-Nikodym property.*

(ii) *The set K has the weak Radon-Nikodym property for the unit interval.*

(iii) *The set K is a set of complete continuity.*

PROOF. (ii) $\Rightarrow$ (iii). Suppose that $0 \in K$. (ii) and the proof of Theorem 1 imply that $K - K$ is a set of complete continuity. Hence, K is a set of complete continuity.

(iii) $\Rightarrow$ (i). (iii) and the proof of Theorem 1 imply that $K - K$ is a set of complete continuity, therefore $K - K$ has the weak Radon-Nikodym property **[10]** and hence, $K \subset K - K$ has the weak Radon-Nikodym property by Corollary 2.

In the definition of a weak Radon-Nikodym set K we required the Pettis kernel of the operator T to have values in K; the following corollary relieves us from this restriction.

COROLLARY 4. *A w^*-compact convex subset K of the dual E^* of a Banach space E has the weak Radon-Nikodym property if and only if every bounded linear operator T: $L^1[0,1] \to E^*$ such that $T(1_A/\lambda(A)) \in K$ is Pettis-differentiable.*

PROOF. One implication is obvious. If every operator T: $L^1[0,1] \to E^*$ such that $T(1_A/\lambda(A)) \in K$ is Pettis-differentiable, then K is a set of complete continuity and hence, K has the weak Radon-Nikodym property by Corollary 3.

COROLLARY 5. *If K has the weak Radon-Nikodym property then every w^*-compact convex subset M of K is the norm-closed convex hull of its extreme points.*

PROOF. Suppose that $0 \in K$. The set $K - K$ has the weak Radon-Nikodym property by Theorem 1. Apply **[11]** to conclude that every w^*-compact convex subset M of $K - K$ (and in particular in K) is the norm-closed convex hull of its extreme points.

Recall **[12]** that a bounded subset M of the dual E^* is weak*-dentable in $(E^*, \sigma(E^*, E^{**}))$, if for every zero neighborhood V in $(E^*, \sigma(E^*, E^{**}))$ there is a weak*-open slice S of M such that $S - S \subset V$ where S is

$$S = \left\{x^* \in M; x^*(x_0) > \sup_{x^* \in M} x^*(x_0) - \alpha\right\}$$

for $x_0 \in E$ and $\alpha > 0$.

The following was shown in **[12]** about a weak*-compact convex subset K of the dual E^* of a Banach space E: every subset M of K is a weak*-dentable in $(E^*, \sigma(E^*, E^{**}))$ if and only if every x^{**} in E^{**} restricted to any weak*-compact subset M of K has a point of continuity on $(M, \text{weak*})$.

Combining the result of **[12]** and Theorem 1 we get

COROLLARY 6. *A weak*-compact convex subset K of the dual E^* of a Banach space E has the weak Radon-Nikodym property if and only if every subset M of K is weak*-dentable in* $(E^*, \sigma(E^*, E^{**}))$.

We finish by asking: *Is the converse of Corollary* 5 *true*? The answer is yes if K is absolutely convex **[11]**.

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