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CONFORMALLY FLAT SPACES AND A PINCHING PROBLEM ON THE RICCI TENSOR

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ABSTRACT. Recent results of S. I. Goldberg on conformally flat manifolds and hypersurfaces of Euclidean space are extended.

1. Introduction. By applying S.-T. Yau's "maximum principle", S. Goldberg [3] proved that an n-dimensional, $n \geqslant 3$, conformally flat Riemannian manifold with constant scalar curvature R whose Ricci curvature is bounded below, and for which $\sup \operatorname{trace} Q^2 < R^2/(n-1)$, is a space form. A corresponding result for hypersurfaces in Euclidean space was obtained by analogy.

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2. Preliminaries. Let (M, g) be a Riemannian manifold with metric g. The curvature transformation $R(X, Y)$, $X, Y \in T_pM$, where T_pM is the tangent space at $p \in M$, and g are related by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

where ∇_X is the operation of covariant differentiation with respect to X. In terms of a basis $X_1, \ldots, X_n$ of T_pM we set

$$R_{ijkl} = g\big(R(X_i, X_j)X_k, X_l\big), \qquad R_{ij} = \operatorname{trace}\big(X_k \to R(X_i, X_k)X_j\big).$$

We denote the scalar curvature by R, that is, $R = \operatorname{trace} Q$, where Q is the symmetric linear transformation field defined by the Ricci tensor, that is $Q = (R^i_j)$ and $R^i_j = g^{ik}R_{jk}$. The manifold (M, g) is conformally flat if g is conformally related to a locally flat metric. Let M be an n-dimensional ($n \geqslant 3$) conformally flat Riemannian manifold with constant scalar curvature, then the following formula may be found in [3]:

$$\frac{1}{2}\Delta \operatorname{trace} Q^2 = \frac{n}{n-2}\operatorname{trace} Q^3 - \frac{2n-1}{(n-1)(n-2)} R \operatorname{trace} Q^2 + \frac{R^3}{(n-1)(n-2)} + g(\nabla Q, \nabla Q).$$

Put $S = Q - R_I/n$, $I =$ identity. Then $\operatorname{trace} S^2 \geqslant 0$ with equality holding if and only if M is an Einstein space. Obviously $\operatorname{trace} S^2 = \operatorname{trace} Q^2 - R^2/n$, and since R

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is constant we get $\Delta \operatorname{trace} S^2 = \Delta \operatorname{trace} Q^2$, where Δ is the Laplace operator on M. Repeating the same calculations as in [3] we get, for $f^2 = \operatorname{trace} S^2$,

$$\frac{1}{2}\Delta f^2 \geqslant \sqrt{\frac{n}{n-1}}\, f^2\left(\frac{R}{\sqrt{n(n-1)}} - f\right). \tag{2.1}$$

The tool for the proof of the main result is a slight modification [5, Theorem 1] of the generalized maximum principle proved in [1 or 8], which we state as follows: Let M be a complete, connected Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M and which has no maximum. Then for all $\varepsilon > 0$, there exists a point $P \in M$ such that at P,

(1) $\sup f - \varepsilon < f(P) < \sup f - \varepsilon/2$,

(2) $|\operatorname{grad} f|(P) < \varepsilon$,

(3) $\Delta f(P) < \varepsilon$.

3. Main results. The following lemma is fundamental and may be found in [6].

LEMMA. *Let $a_1, \ldots, a_n$ be real numbers satisfying the inequality*

$$\sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Then for any pair of distinct i and $j = 1, \ldots, n$ we have $a_i a_j > 0$.

THEOREM 1. *Let M be an n-dimensional ($n \geqslant 3$), complete, connected conformally flat Riemannian manifold. If its scalar curvature R is a positive constant and* $\operatorname{trace} Q^2 \leqslant R^2/(n-1)$, *then M is a space form or* $\operatorname{trace} Q^2 = R^2/(n-1)$ *everywhere on M.*

PROOF. Let f^2 be as in §2 above; we distinguish two cases.

Case I. f^2 attains its maximum; then by using E. Hopf's well-known theorem we conclude from (2.1) that $f^2 = \text{constant}$ and thus $f^2 = 0$ or $f^2 = R^2/n(n-1)$ everywhere on M. But then $\operatorname{trace} Q^2 = R^2/n$, that is, M is an Einstein space and thus a space form or $\operatorname{trace} Q^2 = R^2/(n-1)$ everywhere on M.

Case II. f^2 has no maximum. Suppose $\sup f^2 < R^2/n(n-1)$; then from (2.1) and by using the same method as in the proof of [4, Theorem A] we conclude that $f^2 = 0$, that is, M is a space form. Now let $\sup f^2 = R^2/n(n-1)$. Since f^2 attains no maximum we also have $f^2 < R^2/n(n-1)$. We prove that this is not true. Obviously, since $f^2 < R^2/n(n-1)$, we get $\operatorname{trace} Q^2 < R^2/(n-1)$. Applying the lemma for the eigenvalues of the Ricci tensor we conclude that the Ricci curvature is bounded from below; in particular, it is positive. By generalized maximum principle we have that, for any natural number m, there exists a point $P_m \in M$ such that (since $\sup f^2 = R^2/(n(n-1))$)

$$\frac{R^2}{n(n-1)} - \frac{1}{m} < f^2(P_m) < \frac{R^2}{n(n-1)} - \frac{1}{2m}, \tag{3.1}$$

$$\sqrt{\frac{n}{n-1}}\, f^2(P_m)\left(\frac{R}{\sqrt{n(n-1)}} - f(P_m)\right) \leqslant \frac{1}{2}\Delta f^2(P_m) < \frac{1}{2m}. \tag{3.2}$$

From (3.1) we get

$$\left(\frac{R}{\sqrt{n(n-1)}} - f(P_m)\right)\left(\frac{R}{\sqrt{n(n-1)}} + f(P_m)\right) > \frac{1}{2m}$$

or

$$\frac{R}{\sqrt{n(n-1)}} - f(P_m) > \frac{1}{2m\left(R/\sqrt{n(n-1)} + f(P_m)\right)}$$

and thus (3.2) becomes

$$\sqrt{\frac{n}{n-1}}\, f^2(P_m) \cdot \frac{1}{2m\left(R/\sqrt{n(n-1)} + f(P_m)\right)} < \frac{1}{2m}$$

or

$$\sqrt{\frac{n}{n-1}}\, f^2(P_m) < \frac{R}{\sqrt{n(n-1)}} + f(P_m)$$

or

$$f^2(P_m) - \sqrt{\frac{n-1}{n}}\, f(P_m) - \frac{R}{n} < 0. \tag{3.3}$$

From (3.3), since $f(P_m) > 0$, we get

$$f(P_m) < \frac{\sqrt{n-1+4R} + \sqrt{n-1}}{2\sqrt{n}}$$

and thus

$$\sup f \leqslant \frac{\sqrt{n-1+4R} + \sqrt{n-1}}{2\sqrt{n}}.$$

Now, $\sup f = R/\sqrt{n(n-1)}$ and, comparing with the last inequality, we take

$$\frac{R}{\sqrt{n(n-1)}} \leqslant \frac{\sqrt{n-1+4R} + \sqrt{n-1}}{2\sqrt{n}}$$

or

$$2R - (n-1) \leqslant \sqrt{(n-1)^2 + 4R(n-1)}$$

or

$$R \leqslant 2(n-1). \tag{3.4}$$

Now let λ be a positive constant, then the Riemannian manifold $(M, \lambda g)$ has scalar curvature $\bar{R} = R/\lambda$ and satisfies the same assumptions as (M, g). Then we must have, as above,

$$\bar{R} = R/\lambda \leqslant 2(n-1) \quad \text{or} \quad R \leqslant 2\lambda(n-1),$$

which is impossible for $\lambda < R/2(n-1)$. This completes the proof of the theorem.

COROLLARY 1. *Let M be an n-dimensional* ($n \geqslant 3$), *complete, connected conformally flat Riemannian manifold. If its scalar curvature R is a positive constant and* trace $Q^2 < R^2/(n-1)$, *then M is a space form.*

REMARK. If on a conformally flat Riemannian manifold with positive constant scalar curvature R, trace $Q^2 = R^2/(n-1)$ everywhere, then it follows easily [**2**, Theorem 3] that M is a Riemannian product of a space form M_1, with a 1-dimensional Riemannian manifold N, i.e., $M = M_1 \times N$.

Thus we have

THEOREM 1′. *The only n-dimensional* ($n \geqslant 3$), *complete, connected conformally flat Riemannian manifolds with positive constant scalar curvature such that* trace $Q^2 \leqslant R^2/(n-1)$, *are the space forms and the Riemannian products $M_1 \times N$ where M_1 is a space form and N is* 1-*dimensional.*

In a similar manner, we obtain the following extension of a theorem of Okumura [**7**].

THEOREM 2. *Let M be an n-dimensional* ($n \geqslant 3$), *complete, connected hypersurface of Euclidean space E^{n+1}. If the mean curvature H is constant and $S \leqslant n^2H^2/(n-1)$, where S is the square of the second fundamental form, then M is a hyperplane, a hypersphere or a circular cylinder $S^{n-1} \times E$.*

COROLLARY 2. *Let M be an n-dimensional* ($n \geqslant 3$), *complete, connected hypersurface of E^{n+1}. If the mean curvature is constant and $S < n^2H^2/(n-1)$, then M is a hypersphere.*

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