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# MINIMAL ENTROPY FOR ENDOMORPHISMS OF THE CIRCLE

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ABSTRACT. Let $f$ be an endomorphism (continuous map) of the circle which has two periodic points of period $m$ and $n$ respectively such that $m \geq 2$, $n \geq 2$ and $(m,n) = 1$, then topological entropy $h(f) \geq \log \mu_{m,n}$ where $\mu_{m,n}$ is the largest zero of the polynomial $x^{m+n} - x^m - x^n - 1$.

**Introduction.** In [3], L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young gave, among other things, minimal topological entropy for several types of endomorphisms (continuous maps) of the circle which have a fixed point and another periodic point. Main results in [3] will be shown later as Lemmas 2.1 and 2.4 without proof. L. Block, E. M. Coven and Z. Nitecki [2] have given some improved estimates of minimal entropy after [3]. The aim of this paper is to prove the following.

THEOREM B. *Let $f$ be an endomorphism of the circle which has two periodic points of period $m$ and $n$ respectively such that $m \geq 2$, $n \geq 2$ and $(m,n) = 1$. Then topological entropy $h(f) \geq \log \mu_{m,n}$ where $\mu_{m,n}$ is the largest zero of $x^{m+n} - x^m - x^n - 1$.*

Notice that $f$ has no condition on degree. We also give an example of endomorphism which attains the smallest possible entropy $\log \mu_{m,n}$.

Let $R$ denote the real numbers, $Z$ the integers, $N$ the positive integers, and $S = R/Z$ the circle. Let $\pi\colon R \to S$ denote the canonical projection. For simplicity, we will often write $x$ instead of $\pi(x)$ for $0 \leq x < 1$. Let $f\colon S \to S$ be an endomorphism of degree $k$. Choose a lifting $F\colon R \to R$, that is a map such that $\pi F = f\pi$. Liftings exist and are unique up to the addition of an integer. Each lifting satisfies $F(x+1) = F(x) + k$.

DEFINITION. Let $f$ be an endomorphism of degree 1 and $F$ be a lifting of $f$. We define the *rotation number*

$$\rho(F,x) = \limsup_{n\to\infty} \frac{1}{n}(F^n(x) - x)$$

and the *rotation set*

$$\rho(F) = \{\rho(F,x) \colon x \in R\} = \{\rho(F,x) \colon x \in [0,1)\}.$$

Notice that if a different lifting is used, then this simply has the effect of translating the rotation number and set by an integer.

It is known (see [4 or 6]) that $\rho(F)$ is a single point or a closed interval, and if $p/m \in \rho(F)$ and $(p,m) = 1$ then $f$ has a periodic point $a$ of period $m$ with $\rho(F,a) =$

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$p/m$. Conversely if $f$ has a periodic point of period $m$ then there is $l \in Z$ such that $l/m \in \rho(F)$. Moreover it is easy to see that if $F$ is monotone then $\rho(F)$ is a single point.

We remark here that in [1] C. Bernhardt gave relations among rotation set, topological entropy and, so-called, twist number for the set of endomorphisms of degree 1 having one maximum and one minimum, to which the one attaining the smallest entropy in the example belongs.

**1.** Let $f$ be an endomorphism of $S$ of degree 1. By an *orbit* of periodic points of $f$ of period $m$, we mean a set $\{a_0, \dots, a_{m-1}\} = \{f^i(a_0): i = 0, \dots, m-1\}$ such that $f^m(a_0) = a_0$ and $0 \le a_0 < a_1 < \cdots < a_{m-1} < 1$ in $S$. We define $a_k = a_i + u$ in $R$ if $a_i$ is in the orbit of period $m$, $k = mu + i$, $k$, $u$, $i \in Z$ and $0 \le i \le m-1$, and use this notation throughout this paper.

LEMMA 1.1. *Let $f$ have an orbit $\{a_0, \dots, a_{m-1}\}$ of periodic points of period $m \ge 2$ and have no periodic points of period less than $m$. Then $a_k < a_l$ implies $F(a_k) < F(a_l)$ where $a_k$ and $a_l$ are defined as above for $k$ and $l \in Z$.*

PROOF. By definition $a_k < a_l$ if $k < l$. Suppose $F(a_s) > F(a_t)$ for $a_s < a_t$, then there exists $j$ such that $F(a_{j-1}) < F(a_j)$ and $F(a_j) > F(a_{j+1})$. Let $I_k = [a_k, a_{k+1}]$, $k \in Z$, be intervals on $R$. As $F(a_{j-1}) \ne F(a_{j+1})$, there exists two adjacent intervals, say $I_{l-1}$ and $I_l$, such that $F(I_{j-1}) \cap F(I_j) \supset I_l$ and either $F(I_{j-1}) \supset I_{l-1}$ or $F(I_j) \supset I_{l-1}$. Since we have only $m$ different intervals $I_0, \dots, I_{m-1}$ on $S$, a sequence $I_{i_1} \to I_{i_2} \to \cdots \to I_{i_{m-1}}$ of $m-1$ intervals on $S$ such that $f(I_{i_\alpha}) \supset I_{i_{\alpha+1}}$ and $I_{i_1} = \pi(I_l)$, either includes one of $\pi(I_{j-1})$ and $\pi(I_j)$ or has some interval, say $I_r$, twice. The first case gives the sequence $\pi(I_l) \to \cdots \to \pi(I_{j-1})$ or $\pi(I_j) \to \pi(I_l)$ so that we have $f^\sigma(\pi(I_l)) \supset \pi(I_l)$ for some $\sigma \le m-1$. The second case implies $f^\tau(I_r) \supset I_r$ for some $\tau \le m-2$. Therefore we have a periodic point of period less than $m$ in each case, contradicting the hypothesis.

Let $A_m = \{a_0, \dots, a_{m-1}\}$ and $B_n = \{b_0, \dots, b_{n-1}\}$ be two orbits of periodic points of $f$ of period $m$ and $n$ respectively. Assume $\rho(F, a_i) \ne \rho(F, b_j)$ for a lifting $F$ of $f$. Let $\hat{F}: R \to R$ be a mapping such that the graph of $\hat{F}$ is made of line segments connecting $(c, F(c))$ and $(d, F(d))$ where $c$ and $d$ are adjacent points among the set $\{a_k: k \in Z\} \cup \{b_l: l \in Z\}$ in $R$. Let $\hat{f}$ be the endomorphism of $S$, of which $\hat{F}$ is a lifting. In fact $\hat{f}$ is $\pi\hat{F}|[0,1]$ and of degree 1. We call $\hat{F}$ and $\hat{f}$ $\langle m, n\rangle$-*skeletons* of $F$ and $f$ respectively with respect to $A_m$ and $B_n$. By the manner of constructing $\hat{f}$ from $f$, we have $h(\hat{f}) \le h(f)$ (see [5]).

DEFINITION. $\langle m, n\rangle$-skeletons $\hat{F}$ and $\hat{f}$ with respect to $A_m$ and $B_n$ are called *simple* if $\hat{F}$ keeps the order among $\{a_k: k \in Z\}$, and if

(1) $\rho(\hat{F}, a_i) > \rho(\hat{F}, b_j)$ and $\hat{F}$ does not have a local maximum at any $b_j$, or

(2) $\rho(\hat{F}, a_i) < \rho(\hat{F}, b_j)$ and $\hat{F}$ does not have a local minimum at any $b_j$.

Hereafter we often assume the following hypothesis for $f$ and say $f$ satisfies $H_{m,n}$.

($H_{m,n}$): (I) $f$ is an endomorphism of $S$ of degree 1.

(II) $f$ has two orbits of periodic points $A_m = \{a_0, \dots, a_{m-1}\}$ and $B_n = \{b_0, \dots, b_{n-1}\}$ such that $\rho(F, a_0) \ne \rho(F, b_0)$ for a lifting $F$ of $f$.

The next lemma will be used only in the proof of Lemma 1.3.

LEMMA 1.2. *Let $F$ and $G$ be liftings of endomorphisms $f$ and $g$ of degree 1 respectively. Assume that for any $x \in R$ there exists $y \in R$ such that $y \ge x$ and*

*$F(x) \geq G(y)$. Then $\rho^-(F) \geq \rho^-(G)$ where $\rho^-(F)$ denotes the largest lower bound of the rotation set $\rho(F)$.*

PROOF. First we prove by induction on $n$ that for any $x \in R$ there exists $y_n$ such that $y_n \geq x$ and $F^n(x) \geq G^n(y_n)$. For $n = 1$, this is just the assumption of the lemma. Assume that it is true for $n$. Then there exists $u \in R$ such that $u \geq F(x)$ and $F^{n+1}(x) = F^n(F(x)) \geq G^n(u)$. Since there is $y_1$ such that $y_1 \geq x$ and $F(x) \geq G(y_1)$, and $G(x+p) = G(x)+p$ for $p \in Z$, we have $y_{n+1} \in R$ such that $y_{n+1} \geq y_1$ and $G(y_{n+1}) = u$. This $y_{n+1}$ satisfies $y_{n+1} \geq x$ and $F^{n+1}(x) \geq G^n(u) = G^n(G(y_{n+1})) = G^{n+1}(y_{n+1})$, completing the induction.

To prove $\rho^-(F) \geq \rho^-(G)$, it suffices to show $\rho^-(G) \leq k/n$ for any rational number $k/n \geq \rho^-(F)$. Assume $\rho^-(F) \leq k/n$. Then we have either $a \in [0,1]$ such that $F^n(a) = a+k$ or $F^n(x) < x+k$ for any $x \in [0,1]$. In the first case we have $b \in R$ such that $b \geq a$ and $G^n(b) \leq F^n(a) = a+k \leq b+k$. In the second case we have $c$ such that $c \geq 1$ and $G^n(c) \leq F^n(1) < 1+k \leq c+k$. Since $G^n(x+p) = G^n(x)+p$ for $p \in Z$, in both cases, we have either $y \in [0,1]$ such that $G^n(y) = y+k$ or $G^n(x) < x+k$ for any $x$. Therefore we have $\rho^-(G) \leq k/n$, completing the proof.

LEMMA 1.3. *Let $f$ satisfy $H_{m,n}$ where $1 < m \leq n$ and have no periodic points of period less than $m$. Then there exists a piecewise linear endomorphism $g$ of $S$ of degree 1 satisfying the following.*

(1) *$g$ satisfies $H_{m,n'}$ for some $n'$ such that $1 < n' \leq n$,*

(2) *$g$ is a simple $\langle m, n' \rangle$-skeleton of itself,*

(3) *$g$ has no fixed point,*

(4) *$h(g) \leq h(f)$.*

PROOF. We may assume $\rho(F, a_0) > \rho(F, b_0)$, because the case $\rho(F, a_0) < \rho(F, b_0)$ goes similarly. If $\hat{f}$, skeleton of $f$, is simple then we have nothing to prove. Let $\hat{f}$ be not simple. Since $\hat{F}$ preserves the order among $\{a_k : k \in Z\}$ by Lemma 1.1, $\hat{F}$ has a local maximum at some $b_j$. Take, say $c_s$ and $c_t$ from the set $\{a_k : k \in Z\} \cup \{b_l : l \in Z\}$ such that $c_s < b_j < c_t$ and no element of $\{a_k : k \in Z\} \cup \{b_l : l \in Z\}$ is in $(c_s, c_t)$ except $b_j$. We modify the graph of $\hat{F}$ on $[c_s, c_t]$ into the line segment connecting $(c_s, F(c_s))$ and $(c_t, F(c_t))$. By doing similar modification on all the intervals of type $[c_s+u, c_t+u]$, $u \in Z$, we have a mapping $F_1 : R \to R$ whose graph is constructed by these modifications from the graph of $\hat{F}$. If $F_1$ still has a local maximum at some $b_{j'}$, then we do similar modifications around $b_{j'}+u$, $u \in Z$, as above. Since $\{b_0, \ldots, b_{n-1}\}$ is a finite set, eventually we have a mapping $F'$ which does not have a local maximum at any point of $\{b_l : l \in Z\}$. Let $f' = \pi F' | [0,1)$, then $F'$ is a lifting of $f'$ and $f'$ still has the orbit $\{a_0, \ldots, a_{m-1}\}$. Since $F(x) \geq F'(x)$, we have $\rho^-(F) \geq \rho^-(F')$ by Lemma 1.2. Thus we have $\rho(F') \supset [\rho^-(F), \rho(F, a_0)] \supset [\rho(F, b_0), \rho(F, a_0)]$. Therefore we have an orbit of periodic points of $f'$, $\{b'_0, \ldots, b'_{n'}\}$ such that $n' \leq n$ and $\rho(F', b'_0) < \rho(F', a_0)$. Let $\hat{f}'$ and $\hat{F}'$ be the $\langle m, n' \rangle$-skeletons of $f'$ and $F'$ respectively with respect to $\{a_0, \ldots, a_{m-1}\}$ and $\{b'_0, \ldots, b'_{n'-1}\}$. By assumption and the manner of constructing $\hat{f}'$, $\hat{f}'$ has no fixed point. Any local maximum of $F'$ is at some $a_i$ and $F'$ preserves the order among $\{a_k : k \in Z\}$. Moreover $\rho(\hat{F}', b'_0) = \rho(F', b'_0) < \rho(F', a_0) = \rho(\hat{F}', a_0)$. Thus $\hat{f}'$ satisfies (1), (2) and (3). By the manner of constructing $\hat{f}'$ from $f$, $g = \hat{f}'$ also satisfies (4) (see [5]).

Let $f$ satisfy $H_{m,n}$ with respect to $A_m = \{a_0, \ldots, a_{m-1}\}$ and $B_n = \{b_0, \ldots, b_{n-1}\}$. Let $C = \{c_0, \ldots, c_{m+n-1}\} = A_m \cup B_n$ and $0 \le c_0 < \cdots < c_{m+n-1}$ on $S$. Let $c_\varsigma = c_\eta + w$ for $\varsigma = w(m+n) + \eta$, $\varsigma$, $\eta$, $w \in Z$ and $0 \le \eta \le m+n-1$.

Let $I_\eta = [c_\eta, c_{\eta+1}]$, $\eta = 0, \ldots, m+n-2$, and $I_{m+n-1} = [c_{m+n-1}, c_0]$ be intervals on $S$. We say $I_\xi$ *$f$-covers* $I_\eta$ $p$ times if there exist subintervals $K_1, \ldots, K_p$ of $I_\xi$ with pairwise disjoint interiors such that $f(K_i) = I_\eta$ for $i = 1, \ldots, p$.

DEFINITION. An *A-graph of $f$* with respect to $A_m$ and $B_n$ is an oriented graph with vertices $I_1, \ldots, I_{m+n-1}$ such that if $I_\xi$ $f$-covers $I_\eta$ $p$ times but not $p+1$ times then there are $p$ arrows from $I_\xi$ to $I_\eta$. A sequence $I_{i_0} \to \cdots \to I_{i_r}$ in an $A$-graph of $f$ is called a *path* of *length* $r$, and the path is called a *loop* if $I_{i_0} = I_{i_r}$. A loop is called *simple* if $I_{i_\alpha} \ne I_{i_\beta}$ for $0 \le \alpha < \beta < r$.

LEMMA 1.4. *Let $f$ satisfy $H_{m,n}$ where $1 < m$ and $1 < n$, have no fixed points and be the simple $\langle m,n\rangle$-skeleton of itself. Then the A-graph of $f$ with respect to $A_m$ and $B_n$ has three different simple loops $L_1$, $L_2$, and $L_3$ through a vertex, say $J$, of length $l_1$, $l_2$ and $l_3$ such that $l_1 \le m$, $l_2 \le n$ and $l_3 \le m+n$, and the last vertices before $J$ of $L_1$, $L_2$ and $L_3$ are all different.*

PROOF. We may assume $\rho(F, a_0) > \rho(F, b_0)$ for a lifting $F$ of $f$. Let $F$ have a local maximum at $a_i$. Let $b_j$ be the adjacent one of $a_i$ to the right ($a_i < b_j$) among $\{a_k \colon k \in Z\} \cup \{b_l \colon l \in Z\}$. Then $F(a_i) > F(b_j)$ and there exists $a_s$ and $b_t$ such that $F(a_i) \ge a_s > b_t \ge F(b_j)$, $(b_t, F(a_i)] \cap \{b_l \colon l \in Z\} = \varnothing$ and $[b_t, a_s) \cap \{a_k \colon k \in Z\} = \varnothing$. And we note $a_{i+1} = \inf\{a_k \colon a_k > b_j\}$ and $b_{j-1} = \sup\{b_l \colon b_l < a_i\}$. Since $\{a_0, \ldots, a_{m-1}\}$ are periodic and $f$ has no fixed point, there exists $\alpha \le m-1$ such that $F^\alpha(a_s) = a_i + v$ for some $v \in Z$. Since $f$ is simple and $\rho(F, a_0) > \rho(F, b_0)$, $b_l < a_k$ implies $F(b_l) < F(a_k)$. Thus we have $F^\alpha(b_t) < a_i + v$ and so $F^\alpha(b_t) \le b_{j-1} + v$. Let $J = \pi([b_t, a_s])$, then $f^\alpha(J) \supset \pi([b_{j-1}, a_i])$ and $f(\pi([b_{j-1}, a_i])) \supset J$. On the other hand, we have $F^\beta(b_t) = b_j + w$ for some $\beta \le n-1$ and $w \in Z$. Thus $F^\beta(a_s) > b_j + w$ and $F^\beta(a_s) \ge a_{i+1} + w$. Therefore $f^\beta(J) \supset \pi([b_j, a_{i+1}])$ and $f(\pi([b_j, a_{i+1}])) \supset J$.

Now we have two different intervals say $I_p \subset \pi([b_{j-1}, a_i])$ and $I_q \subset \pi([b_j, a_{i+1}])$ such that $f^\alpha(J) \supset I_p$, $f(I_p) \supset J$, $f^\beta(J) \supset I_q$ and $f(I_q) \supset J$ where $0 < \alpha \le m-1$ and $0 < \beta \le n-1$. Furthermore we have another interval $I_r = \pi([a_i, b_j])$ satisfying $f(I_r) \supset J$. As $\rho(F, a_s) \ne \rho(F, b_t)$, we have $f^\sigma(J) \supset I_r$ for some $\sigma \in N$ large enough. Since we have only $m+n$ different intervals on $S$, there exists $0 < \gamma \le m+n-1$ such that $f^\gamma(J) \supset I_r$. Therefore we have three different loops of length $\alpha+1$, $\beta+1$ and $\gamma+1$ from $J$ to $J$. If any of them is not simple, then it is easy to see we can get an even shorter loop which is simple and has the same last vertex before $J$. Consequently the $A$-graph of $f$ has three simple loops which satisfy the statement of the lemma.

LEMMA 1.5. *Let $f$ satisfy $H_{m,n}$ and the A-graph of $f$ with respect to $A_m$ and $B_n$ have three simple loops $L_1$, $L_2$ and $L_3$ through a vertex $J$ of length $l_1$, $l_2$ and $l_3$ respectively where $l_1 \le l_2 \le l_3$. Assume the last vertices $I_p$, $I_q$ and $I_r$ of $L_1$, $L_2$ and $L_3$ before $J$ are all different, and that $L_1$, $L_2$ and $L_3$ are the shortest loops of all the loops through $J$ whose last vertices are $I_p$, $I_q$ and $I_r$ respectively. Then the A-graph of $f$ has a subgraph which has just three simple loops $L_1$, $L_2'$ and $L_3'$ all through $J$ of length $l_1$, $l_2$ and $l_3$ respectively such that the last vertices of $L_2'$ and $L_3'$ before $J$ are $I_q$ and $I_r$ respectively.*

PROOF. If $L_2$ does not intersect with $L_1$ at vertices other than $J$, then let $L_2' = L_2$. If $L_2$ intersects with $L_1$, let $I_s \ne J$ be the last vertex of $L_2$ that is also on $L_1$. Let $L_2'$ be the loop which is on $L_1$ from $J$ to $I_s$ and is on $L_2$ from $I_s$ to $J$. Because of the shortness of $L_1$ and $L_2$, the length of $L_2'$ is $l_2$. If $L_3$ does not intersect with neither $L_1$ nor $L_2'$ outside of $J$, let $L_3' = L_3$. If $L_3$ intersects with $L_1$ or $L_2'$, let $I_t \neq J$ be the last vertex of $L_3$ that is also on $L_1$ or $L_2'$. Let $L_3'$ be the loop which is on either $L_1$ or $L_2'$ from $J$ to $I_t$ depending on whether $I_t$ is on $L$ or on $L_2'$, and is on $L_3$ from $I_t$ to $J$. Again by the shortness of $L_1$, $L_2'$ and $L_3$, the length of $L_3'$ is $l_3$. Therefore the subgraph consisting of $L_1$, $L_2'$ and $L_3'$ satisfies the requirement.

Let $C$ be an $A$-graph of $g$ and $s$ be the number of vertices of $G$. Following [**3**], we associate to $G$ an $s \times s$ matrix $M = (m_{ij})$ such that $m_{ij} =$ (number of arrows from $I_i$ to $I_j$). We call the logarithm of the spectral radius (i.e. of the largest eigenvalue) of $M$ the *entropy of* $G$ and denote it by $h(G)$.

LEMMA 1.6 [**3**, LEMMA 1.5]. *If $G$ is an $A$-graph of $g$, then $h(G) \leq h(g)$.*

Since $h(G)$ is the limit of $\frac{1}{n}\log$ (sum of entries of $M^n$) the following is trivial.

LEMMA 1.7. *Let $G'$ be a subgraph of $G$, then $h(G') \leq h(G)$.*

As a special case of Theorem 1.7 of [**3**] we have the following.

LEMMA 1.8. *Let $G$ be a graph consisting of three different simple loops $L_1$, $L_2$ and $L_3$ all through a vertex $J$ of length $l_1$, $l_2$ and $l_3$ respectively. Assume $G$ has no other simple loops. Let $M$ be the matrix associated to $G$. Then the characteristic polynomial of $M$ is $(-1)^s(x^s - x^{s-l_1} - x^{s-l_2} - x^{s-l_3})$, where $s$ is the number of vertices of $G$.*

The next lemma is easily proved by direct calculation or by application of Lemma 1.8 of [**3**].

LEMMA 1.9. *Let $m$, $n$, $l_1$, $l_2$ and $l_3$ be positive integers such that $l_1 \leq l_2 \leq l_3$, $l_1 \leq m$, $l_2 \leq n$ and $l_3 \leq m+n$. Let $\mu_{m,n}$ and $\nu$ be the largest zeros of $x^{m+n} - x^m - x^n - 1$ and of $x^{l_3} - x^{l_3-l_1} - x^{l_3-l_2} - 1$ respectively. Then*
*(1) $\mu_{m,n} \leq \nu$,*
*(2) if $1 \leq m' \leq m$ and $1 \leq n' \leq n$, then $\mu_{m,n} \leq \mu_{m',n'}$.*

THEOREM A. *Let $f$ satisfy $H_{m,n}$. Then $h(f) \geq \log \mu_{m,n}$ where $\mu_{m,n}$ is the largest zero of $x^{m+n} - x^m - x^n - 1$.*

PROOF. By Lemma 1.3 and 1.9(2), it suffices to prove $h(g) \geq \log \mu_{m,n'}$ for $g$ satisfying (1), (2) and (3) of Lemma 1.3. Let $G$ be the $A$-graph of $g$ with respect to the two orbits of period $m$ and $n'$. Then, by Lemmas 1.4 and 1.5, $G$ has a subgraph $G'$ which has $s$ vertices and just three simple loops $L_1$, $L_2$ and $L_3$ all through $J$ of length $l_1$, $l_2$ and $l_3$ respectively such that $l_1 \leq l_2 \leq l_3$, $l_1 \leq \min\{m, n'\}$, $l_2 \leq \max\{m, n'\}$ and $l_3 \leq m + n'$. By Lemma 1.8, $h(G') = \log \nu$ where $\nu$ is the largest zero of $x^s - x^{s-l_1} - x^{s-l_2} - x^{s-l_3}$, i.e. that of $x^{l_3} - x^{l_3-l_1} - x^{l_3-l_2} - 1$. By Lemma 1.9(1), we have $\mu_{m,n'} \leq \nu$ and by Lemmas 1.6 and 1.7,

$$h(g) \geq h(G) \geq h(G').$$

Therefore $h(g) \geq \log \mu_{m,n'}$.

**2.** In this section we shall prove Theorem B stated in the introduction. First we need several lemmas from [**3**].

LEMMA 2.1 [**3**, THEOREM 3.2]. *If $f$ is an endomorphism of the circle of $|\deg f| \leq 1$, $F$ is a lifting of $f$ and $F$ has a periodic point of period $m > 1$, then $h(f) \geq (\log \lambda_p)/2^k$ if $m = 2^k \cdot p$, $p$ is odd and $p > 1$, where $\lambda_p$ is the largest zero of $x^p - 2x^{p-2} - 1$.*

LEMMA 2.2 [**3**, PROPOSITION 3.4]. *Let $\deg f = -1$ and let $x \in R$ be a point such that $\pi(x)$ is a periodic point of $f$ of period $m$, $m$ odd. Then there exists a lifting $F$ of $f$ such that $x$ is a periodic point of $F$ of period $m$.*

The next lemma is trivial.

LEMMA 2.3. *The largest zero $\lambda_m$ of $x^m - 2x^{m-2} - 1$ is larger than the largest zero $\mu_{m,n}$ of $x^{m+n} - x^m - x^n - 1$ where $m \geq 2$ and $n \geq 2$.*

The following is one of the main theorems of [**3**].

LEMMA 2.4 [**3**, THEOREM 3.9]. *Let $f$ be an endomorphism of $S$ of degree 1. Let $f$ have a fixed point $x$ and a periodic point $y$ of period $n > 1$ such that $\rho(F,x) \neq \rho(F,y)$ where $F$ is a lifting of $f$. Then $h(f) \geq \log \mu_n$ where $\mu_n$ is the largest zero of the polynomial $x^{n+1} - x^n - x - 1$.*

PROOF OF THEOREM B. (1) $\deg f = 1$. If $f$ has no fixed point, then this theorem is a special case of Theorem A which we have proved in §1. If $f$ has a fixed point $x$, then we may choose a lifting $F$ of $f$ such that $\rho(F,x) = 0$. Let $a$ and $b$ be periodic points of $f$ of period $m$ and $n$ respectively such that $m \geq 2$, $n \geq 2$ and $(m,n) = 1$. If $\rho(F,a) = \rho(F,b) = 0$, then $a$ and $b$ are also periodic points of $F$ of period $m$ and $n$. As $(m,n) = 1$, we may assume $m$ is odd and $m \geq 3$. Then, by Lemmas 2.1 and 2.3 we have $h(f) \geq \log \lambda_m > \log \mu_{m,n}$. If either $\rho(F,a) \neq 0$ or $\rho(F,b) \neq 0$, then by Lemma 2.4, $h(f) \geq \log \mu_m$ or $h(f) \geq \log \mu_n$. On the other hand, by Lemma 1.9(2), $\min\{\mu_m, \mu_n\} \geq \mu_{m,n}$. Therefore $h(f) \geq \log \mu_{m,n}$.

(2) $\deg f = -1$. We may assume $m$ is odd and $m \geq 3$. By Lemmas 2.2, 2.1 and 2.3, $h(f) \geq \log \lambda_m > \log \mu_{m,n}$.

(3) $\deg f = 0$. We may assume $m$ is odd and $m \geq 3$. In this case, given any periodic point of $f$, $F$ has a periodic point of the same period (see [**3**, Proposition 3.3]). therefore, by Lemmas 2.1 and 2.3, we have $h(f) \geq \log \lambda_m > \log \mu_{m,n}$.

(4) $|\deg f| \geq 2$. $h(f) \geq \log |\deg f| \geq \log 2 > \log \mu_{m,n}$.

Finally we give an example to show our estimate of the smallest entropy in Theorem B is the best possible.

EXAMPLE. Let $m$ and $n$ be positive integers such that $1 < m < n$ and $(m,n) = 1$. Then there exist $p$ and $q \in N$ such that $p < m$, $q < n$ and $np - mq = 1$.

Let us define $F \colon R \to R$ as the following.

$$F(x) = \begin{cases} x + \dfrac{q}{n}, & 0 \leq x \leq 1 - \dfrac{1}{n}, \\ \dfrac{2m-1}{m-1}\left(x - \dfrac{n-1}{n}\right) + \dfrac{q-1}{n} + 1, & 1 - \dfrac{1}{n} \leq x \leq 1 - \dfrac{1}{mn}, \\ (1-m)(x-1) + \dfrac{q}{n} + 1, & 1 - \dfrac{1}{mn} \leq x \leq 1 \end{cases}$$

and $F(x+k) = F(x) + k$ for $k \in Z$.

This is a piecewise linear function obtained by modifying $x + q/n$ slightly to assume a maximum $1+(q+1)/n-1/mn$ at $x = 1-1/mn$. Let $f(x) = \pi F(x)|[0,1]$, then it is an endomorphism of $S$ of degree 1 and has periodic points $\pi(jq/n)$, $j = 0,\dots,n-1$, of period $n$ and $\pi((iq+1)/n-1/mn)$, $i = 1,\dots,m$, of period $m$. Let $G$ be the $A$-graph of $f$ with respect to these periodic points of period $m$ and $n$ shown above. Then it is not difficult to see $h(f) = h(G) = \log \mu_{m,n}$.

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