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COMMON FIXED POINTS FOR A CLASS OF COMMUTING MAPPINGS ON AN INTERVAL

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ABSTRACT. Let C be a family of continuous commuting functions of an interval I into itself. If each function, except for possibly one, has an interval $[a, b]$, $a \leqslant b$, for its set of fixed points or does not have periodic points except fixed ones, then it is shown that C has a common fixed point. This result generalizes a previous theorem of T. Mitchell.

1. Introduction. T. Mitchell [2] proved by means of topological semigroup methods that a family F of commuting continuous self-maps of an interval such that for all $f \in F$, the iterates of f form an equicontinuous family except for one possible exception, have a common fixed point. This was a generalization of W. Boyce's result [1], where only two functions were used. Both employed the techniques developed by A. Shields in [3].

In this note a larger class of functions is considered which has the common fixed point property and contains properly the class F considered by Mitchell. This result is obtained by elementary means, and no use is made of topological semigroup methods.

2. Notation and terminology. All functions considered here are assumed to be continuous from the interval $I = [u, v]$ to itself.

Denote by F_f and P_f the set of fixed and periodic points of f respectively, and by $L_f(x)$ the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. Use is made of a result by Schwartz [4] to the effect that $L_f(x) \cap \overline{P}_f \neq \varnothing \ \forall x \in I$.

Define the classes of functions:

$$A = \{f\colon I \to I \mid F_f = [a_f, b_f],\, a_f \leqslant b_f\},$$
$$B = \{f\colon I \to I \mid P_f = F_f\}.$$

A class of functions C is said to be an H-class if $C = C' \cup \{h\}$ where C' is any subset of $A \cup B$ composed of commuting functions and h is any function which commutes with the elements of C'.

A class of functions D is said to be an F-class if $D = D' \cup \{h\}$ where D' is any family of functions such that the iterates of each element of D' form an equicontinuous family, and h is any function that commutes with the elements of D'.

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3. Results.

THEOREM 1. *There is a common fixed point for every H-class in I.*

PROOF. Let C be an H-class, and C_1 a finite subset of C. C_1 can be written as

$$C_1 = \{a_1, \ldots, a_n\} \cup \{h\} \cup \{b_1, b_2, \ldots, b_m\}$$

where $a_i \in A$, $i = 1, \ldots, n$, h is a possible arbitrary function that commutes with the elements of C', $b_j \in B$, $j = 1, \ldots, m$. Since F_{a_i} is an interval and the a_i's commute, $\bigcap_{i=1}^n F_{a_i}$ is an interval $[a, b]$. Also by the commutativity of h with the a_i's, h takes $[a, b]$ into $[a, b]$, and so it must have a fixed point $z \in [a, b]$. Now $\{b_1^n(z)\}_{n=0}^\infty$ has a limit point $z_1 \in F_{b_1}$ since $P_{b_1} = F_{b_1}$ and F_{b_1} is a closed set. But $z_1 \in \bigcap_{i=1}^n F_{a_1} \cap F_h$ because b_1 takes $\bigcap_{i=1}^n F_{a_1} \cap F_h$ into itself. Similarly $\{b_j^n(z_{j-1})\}_{n=0}^\infty, j = 2, \ldots, m$, has a limit point z_j which is fixed for $b_j, \ldots, b_1$, h, $a, \ldots, a_n$; thus $\bigcap F_f \neq \varnothing$ $\forall f \in C_1$. Now since I is compact, $\bigcap F_f \neq \varnothing$ $\forall f \in C$. The cases where C contains no such h, $C \cap A = \varnothing$, or $C \cap B = \varnothing$ are clear from the above proof.

THEOREM 2. *Let f be a function such that its iterates form an equicontinuous family. Then*

(1) $f \in A$,

(2) $f \in B$ *if F_f is nondegenerate.*

PROOF. If F_f is a singleton, we are done. So suppose there are $a, b \in F_f$, $a < b$ and $\forall x \in (a, b)$, $x \notin F_f$. Then $f(x) > x$ or $f(x) < x$. Assume $f(x) > x$ $\forall x \in (a, b)$ (the case $f(x) < x$ is done similarly).

We consider two cases.

(i) $f(x) < b$ $\forall x \in (a, b)$,

(ii) $\exists x \in (a, b) \ni f(x) \geqslant b$.

Case (i). $\forall x \in (a, b)$, $\{f^n(x)\}_{n=0}^\infty \to b$ and so the family of iterates of f cannot be equicontinuous at a.

Case (ii). Let z be the smallest point in (a, b) with $f(z) = b$. There is a sequence $\{x_n\}$ in $(a, z]$ such that $\{x_n\} \to a$, $x_1 = z$ and $f(x_n) = x_{n-1}$. Thus $f^n(x_n) = b$ for $n = 1, 2, \ldots$ and the family of iterates of f cannot be equicontinuous at a also. This contradiction establishes part (1).

For part (2), let $F_f = [a, b]$ with $a < b$. Suppose $f^n(x) = x$ for some n and some $x \in [u, a)$ (the case $x \in (b, v]$ is similar). Applying part (1) to the function f^n, we obtain $f^n(y) = y$ $\forall y \in [x, a]$. But since $f(y) > y$ $\forall y \in [u, a)$, and $f(a) = a$, we may choose $y \in (x, a)$ close enough to a so that either $y < f^n(y) < a$ or $y < f^m(y) \in [a, b]$ for some $m \leqslant n$. Then $f^n(y) > y$, a contradiction.

For $I = [0, 1]$ the function $f(x) = -x + 1$ shows that the condition that F_f be a nondegenerate interval is necessary. The class of functions $\{x, x^2, x^3, \ldots\}$ is an H-class that is not an F-class on $[0, 1]$.

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