PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

THE NONREALIZABILITY OF MODULAR RINGS OF POLYNOMIAL INVARIANTS BY THE COHOMOLOGY OF A TOPOLOGICAL SPACE

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ABSTRACT. Let $G < \mathrm{GL}(n; \mathbf{F}_p)$ be a p-group, p an odd prime, and $R^* := \mathbf{F}_p[x_1, \ldots, x_n]^G$ the ring of invariants. The purpose of this note is to prove that in the case where R^* is a graded polynomial algebra, where $\deg x_1 = \cdots = \deg x_n = 2$, then there is no space X such that $H^*(X; \mathbf{F}_p) \simeq R^*$. This complements the work of Clark and Ewing [3] and Adams and Wilkerson [1] on the case $p \nmid [G; 1]$.

The purpose of this note is to record a sort of complement to the theorems of Clark and Ewing [3] and Adams and Wilkerson [1]. In the preceding cited articles it is shown that every polynomial algebra over the Galois field $\mathbf{F}_p$, generated by elements of degrees relatively prime to p, which is an unstable algebra over the mod p Steenrod algebra, arises as the ring of invariants of a finite group of order prime to p which is generated by pseudoreflections, and is realized as the cohomology algebra of at least one topological space. For p-groups whose rings of invariants are polynomial the situation is quite different. We have

PROPOSITION. *Let p be an odd prime and $G < \mathrm{GL}(n; \mathbf{F}_p)$ be a p-group. Suppose that the ring of invariants $R^* := \mathbf{F}_p[x_1, \ldots, x_n]^G$ is a polynomial algebra, where $x_1, \ldots, x_n$ all have degree* 2. *Then R^* cannot arise as the $\mathbf{F}_p$ cohomology of a space.*

REMARKS. (1) Since $x_1, \ldots, x_n$ all have degree 2 there is a unique structure of unstable algebra over the Steenrod algebra on $\mathbf{F}_p[x_1, \ldots, x_n]$, which in turn induces an analogous structure on R^*.

(2) There are many examples of p-groups satisfying the hypothesis of the proposition, for example, the upper triangular matrices in $\mathrm{GL}(n; \mathbf{F}_p)$. For $n = 2$ the corresponding ring of invariants is $\mathbf{F}_p[\rho_1, \rho_2]$ where one may choose $\rho_1 = x_1$, $\rho_2 = x_2^p - x_2^{p-1}x_1$. The action of the Steenrod algebra is determined by unstability and the equation $P^1\rho_2 = \rho_1^{p-1}\rho_2$.

(3) Recently, Nakajima [5] has characterized the p-groups $G < \mathrm{GL}(n; \mathbf{F}_p)$ with a polynomial ring as invariants.

PROOF OF PROPOSITION. Write $R^* = \mathbf{F}_p[\rho_1, \ldots, \rho_n]$ where $\deg \rho_1 \leqslant \cdots \leqslant \deg \rho_n$. Let $\rho_1, \ldots, \rho_k$ be the generators of degree 2. (There is always at least one such. To see this set $V := \oplus_n \mathbf{F}_p$ and note that $V - V^G$ is a union of orbits consisting of more than one point in each orbit. Since G is a p-group this means $|V - V^G| \equiv 0(p)$

Received by the editors October 21, 1981.
1980 *Mathematics Subject Classification.* Primary 55P99; Secondary 55P45, 55R35.

where $|\ |$ denotes the cardinality of the underlying set. But $|V| = p^n$ and thus $|V^G| \equiv 0(p)$, so $V^G \neq \{0\}$.) The elements $\rho_1, \ldots, \rho_k$ are then a basis for V^G. Suppose X is a topological space such that $H^*(X; \mathbf{F}) \simeq R^*$. then $\rho_1, \ldots, \rho_k$ is the reduction of integral classes, so there is a map

$$f\colon X \to \times_k \mathbf{C}P(\infty)$$

inducing an isomorphism on $H^2(\ ; \mathbf{F}_p)$. Let Y be the fiber of f. An easy spectral sequence argument shows (eg. [**6**, 3.1]) that

$$H^*(Y; \mathbf{F}_p) \simeq \mathbf{F}_p[\rho_{k+1}, \ldots, \rho_n].$$

From [**2**, Example 8, p. 139] it follows that

$$p^s = |G| = \prod_{i=1}^{n-k} \left(\frac{\deg \rho_{k+i}}{2} \right)$$

and thus $\frac{1}{2} \deg \rho_{k+i}$ is a power of p for $i = 1, \ldots, n-k$. Suppose $\deg \rho_{k+1} = 2p^{r+1}$, $r \geqslant 0$. Then by [**4**]

$$(*) \qquad \rho_{k+1}^p = P^{p^{r+1}} \rho_{k+1} = \sum_{k=1}^{r} a_{k_i} \Psi_i(\rho_{k+1}) + b_k \mathfrak{R}(\rho_{k+1})$$

(the Γ_j terms not figuring in the formula since they are of odd degree), where $\deg \mathfrak{R} = 4(p-1)$ and $\deg \Psi_i = 2p^i(p-1)$. So

$$\deg \Psi_i(\rho_{k+1}) = 2p^{r+1} + 2p^i(p-1) \not\equiv 0 \mod 2p^r,$$

$$\deg \mathfrak{R}(\rho_{k+1}) = 2p^{r+1} + 4(p-1) \not\equiv 0 \mod 2p^r.$$

But $H^i(Y; \mathbf{F}_p) = 0$: $2p^{r+1} \leqslant i \leqslant 2p^{r+2}$ and $i \not\equiv 0 \mod 2p^{r+1}$. Whence from $(*)$ we get $\rho_{k+1}^p = 0$ which is a contradiction. $\square$

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