COUNTING FIXED POINTS

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ABSTRACT. In this paper, we develop a machine which enables us to predict, in many cases, the exact number of fixed points of a local diffeomorphism. Though much more general, our technique applies in particular to locally expansive maps on compact, connected, orientable differentiable manifolds.

Introduction. The Lefschetz number has often been interpreted as a count of the "algebraic number" of fixed points of a mapping. The purpose of this note is to show that, in many cases, it can be used to predict the exact number of fixed points.

Although our technique is much more general, it applies in particular to various types of locally expansive maps, and as a consequence we obtain the following

THEOREM. *Let M be a compact, connected, orientable, differentiable manifold, and suppose $f: M \to M$ is a differentiable expanding map. Then the number of fixed points of f is the absolute value of the Lefschetz number of f.*

We conclude with an example showing that the orientability hypothesis is necessary.

Definitions. The notions of differentiable manifolds and differentiable maps, the tangent space TM_p at a point p, the derivatives $Df_p: TM_p \to TN_{f(p)}$ induced by a differentiable map $f: M \to N$ between differentiable manifolds, local diffeomorphisms, and orientability will be defined à la [JM] or [G/P]. By "differentiable" we shall always mean "at least C^1."

In particular, a manifold may be thought of as a subset of some big Euclidean space, so that the tangent space has a natural interpretation as a linear subspace. In this context, a differentiable map is a *local expansion* provided there is a number $k > 1$ so that $\|Df_p(h)\| \geq k\|h\|$, where $\|\ \|$ represent the usual Euclidean norm. In [MS], Shub defines a C^1 map $f: M \to M$ to be *expanding* if there is a $c > 0$ and a $k > 1$ such that $\|D(f^m)_x(h)\| \geq ck^m\|h\|$ for all positive integers m. Assuming M is compact, this a priori weaker notion has the advantage of being independent of the embedding. It will be seen that our techniques apply with equal ease to both types of expansive maps, as well as many other local diffeomorphisms. (For the connections between various types of expansive maps, see [WR]. The fixed point theory of such maps has been examined in [IR1, IR2, HR, GJ, SL and H/R].)

Outline of proof. *Step* 1. Suppose that M is a compact, connected differentiable manifold of dimension m, and $f: M \to M$ is a differentiable map having only finitely many fixed points. The Lefschetz Theorem asserts that the Lefschetz number of f, $L(f)$, which is a homotopy invariant, is $(-1)^m \Sigma$ sign$[\det(Df_x - I)]$, the summation taken over all of the (finitely many) fixed points of f.

(Local expansions and expanding maps have only finitely many fixed points since 1 is clearly not an eigenvalue of Df_x at a fixed point x.)

Therefore, to show that the number of fixed points of f is the absolute value of the Lefschetz number, it is necessary and sufficient to show that if x_1 and x_2 are two fixed points of f, then $\det(Df_{x_1} - I)$ and $\det(Df_{x_2} - I)$ have the same sign.

I tried for a long time to "go continuously" from $Df_{x_1} - I$ to $Df_{x_2} - I$, but was unable to make sense out of this. That I was doomed to failure, even if f is a diffeomorphism, is shown by the following example, which of course is not a local expansion.

Let $g: [-1, 1] \to [-1, 1]$ be defined by $g(t) = \frac{1}{2}(t^3 + t)$. And let f map the unit circle in the complex plane, S^1, to itself be defined by $f(\exp(\pi t)) = \exp(\pi g(t))$, $t \in [-1, 1]$. Then f is a diffeomorphism, f has exactly two fixed points (namely 1 and -1), but since f is homotopic to the identity, $L(f) = \chi(S^1) = 0$, where χ represents the Euler characteristic. (It is interesting to note that there is a sense in which Df "sort of" has eigenvalue 1. Namely, by lifting f to a map F of the universal covering space, $\mathbf{R}$, all of the tangent spaces become identified with a single copy of $\mathbf{R}$, and the lifted map has derivative $F'(x) = Df_x$ which attains the value 1. See [$\mathbf{R/R}$].)

Thus we must go by a more circuitous route.

Step 2. Suppose D is a linear transformation from a finite dimensional real vector space to itself, and suppose both D and $D - I$ are isomorphisms. Then we will show that a necessary and sufficient condition for $\det D$ and $\det(D - I)$ to have the same sign is that the number of eigenvalues of D which lie in the open interval $(0, 1)$, counted with multiplicity, is even. In particular, if there are *no* eigenvalues in the interval $(0, 1)$, then $\det D$ and $\det(D - I)$ have the same sign. Hence, since derivatives of local expansions obviously have no eigenvalues in this interval, we get that sign$[\det(Df_x)] = $ sign$[\det(Df_x - I)]$ at a fixed point of a local expansion. (This is a special case of a result of Leray and Schauder [$\mathbf{L/S}$].)

Consider $p(t) = \det(D - tI)$ for $t \in [0, 1]$. Then $p(0) = \det(D)$ and $p(1) = \det(D - I)$, and these are both nonzero by hypothesis. Of course $p(t)$ is a polynomial—in fact it is $\pm$ the characteristic polynomial of D—and a root of $p(t)$ is precisely an eigenvalue of D. Now, if λ is an eigenvalue of odd multiplicity, then $p(t)$ changes sign in a small neighborhood of λ, and if λ is an eigenvalue of even multiplicity, then $p(t)$ does not change sign near λ. Thus if the number of eigenvalues of D in $(0, 1)$ counting multiplicity is even, then $p(0)$ and $p(1)$ have the same sign, and conversely.

(In the example given at the end of Step 1, note that for one of the fixed points, namely 1, Df_x had an eigenvalue in the interval $(0, 1)$, namely $\frac{1}{2}$.)

It is worth noting that the hypothesis of being a local expansion is much, much stronger than the hypotheses of Step 2. For example, Step 2 applies to $D = \begin{bmatrix} 1/ & 0 \\ 0 & 1/2 \end{bmatrix}$
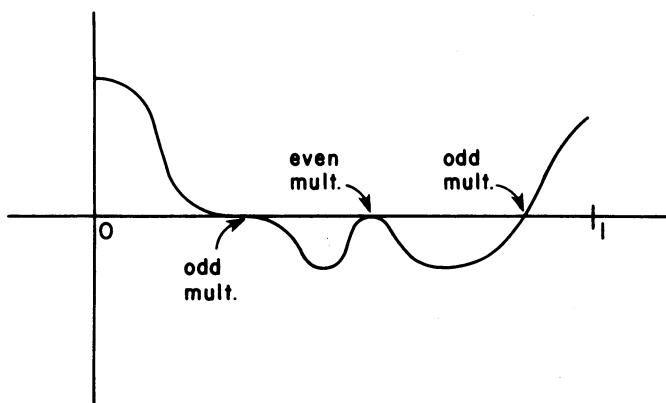
and $D = [\begin{smallmatrix} 0 & -1/2 \\ 1/2 & 0 \end{smallmatrix}]$ both of which are contractions (!), the second of which has complex eigenvalues, and to $D = [\begin{smallmatrix} 0 & -2 \\ 1/2 & 0 \end{smallmatrix}]$ which also has complex eigenvalues, but is "hyperbolic".

Step 3. In the previous step, we have seen that, very often, sign[det($Df_x - I$)] equals sign[det(Df_x)]. While it was difficult to go from one fixed point to another and keep track of sign[det($Df_x - I$)], it is not difficult to keep track of sign[det(Df_x)], at least when M is orientable. We will show the following: Suppose M is a (compact) connected orientable differentiable manifold and $f: M \to M$ is a local diffeomorphism. If x_1 and x_2 are fixed points of f, then sign[det(Df_{x_1})] = sign[det(Df_{x_2})]. Of course, local expansions from a manifold to itself are local diffeomorphisms.

The idea is this. The sign [det(Df_{x_j})] determines whether Df_{x_j} is orientation preserving or reversing (1 if preserving, −1 if reversing) as a function from TM_{x_j} to itself. This, in turn, determines the local behavior of f. But if M is orientable, we can extend that definition by *defining* Sign[x] to be 1 or −1 depending upon whether Df_x is orientation preserving or reversing. (See [**JM**, pp. 26–27].) Now if f is a local diffeomorphism and M is connected, it is easy to see that x cannot change Sign, and thus if x_1 and x_2 are fixed points, det(Df_{x_1}) and det(Df_{x_2}) have the same sign. (Cf. [**G/P**, p. 104].)

EXAMPLE. We now outline an example showing that the orientability hypothesis is necessary. Local expansions are nontrivial covering projections, so if a compact manifold admits a local expansion to itself, it must have zero Euler characteristic. Thus a candidate for a minimal counterexample is the Klein bottle.

Consider the map from the Klein bottle to itself "induced" by the linear map $3I$. (The fixed points are indicated as dots.)
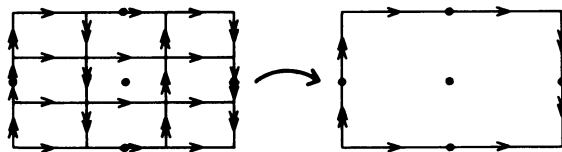
This map is expanding. It has 4 fixed points. And the Lefschetz number is $L = 1 - 3 = -2$. "Thus, for this map, the number of fixed points does not equal the absolute value of the Lefschetz number."

Some questions. 1. Does this result generalize, in particular to PL manifolds, manifolds-with-boundary, or topological groups? What if M is not compact?

2. Suppose $M = D^m$, the m-dimensional disk in $\mathbf{R}^m$, and $f: M \to M$ is differentiable and Df_x never has 1 as an eigenvalue. Does f have a *unique* fixed point? (Here it is easy to make sense of going continuously from $Df_{x_1} - I$ to $Df_{x_2} - I$ without the necessity of Steps 2 and 3.) My guess is yes. Results of Kellogg [**RK**] and later Smith and Stuart [**S/S**] imply a positive answer to this question under the additional assumption that f has no fixed points on the boundary. See [**R/R**] for some related results.

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