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GEOMETRIC REALIZATION OF $\pi_0 \varepsilon(M)$

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ABSTRACT. Let M be a closed flat Riemannian manifold, $\varepsilon(M)$ the group of self homotopy equivalences of M. Then there exists a subgroup $A_1(M)$ of $\mathrm{Aff}(M)$ such that the natural homomorphism of $A_1(M)$ into $\pi_0\varepsilon(M)$ is a surjection with kernel a finite abelian group. Furthermore, this kernel can be identified with the structure group of the Calabi fibration.

This note is concerned with the realization problem of the group $\pi_0\varepsilon(M)$ of homotopy classes of self homotopy equivalences of M as a group of affine diffeomorphisms for closed flat Riemannian manifolds M. We assume familiarity with **[L-R1, 2]** and retain the notations and definitions given there. In **[L-R1]** it is shown that a finite abstract kernel $G \to \mathrm{Out}\,\pi_1 M \cong \pi_0\varepsilon(M)$ can be realized as a group of affine diffeomorphisms of M if and only if it admits an admissible group extension, i.e., if and only if there is a group extension E of $\pi_1 M$ by G realizing the abstract kernel, which is admissible (the centralizer of $\pi_1 M$ in E, $C_E(\pi_1 M)$, is torsion-free). See also **[Z-Z]**.

When the center of $\pi_1 M$ is nontrivial, this condition is not automatically satisfied. In fact, for each n, there exist a closed flat manifold M and a subgroup of $\mathrm{Out}\,\pi_1 M$, isomorphic to $\mathbf{Z}_n$, which cannot be realized as any group of homeomorphisms of M. See **[L-R2]** for more details. Therefore, the next best thing for the realization problem would be finding an inflation of such a subgroup of $\pi_0\varepsilon(M)$ by a finite group, keeping the size of homotopy classes fixed, which can be realized. In fact, this can be done uniformly.

MAIN THEOREM. *Let M be a closed flat Riemannian manifold. Then there exists a subgroup $A_1(M)$ of* $\mathrm{Aff}(M)$ *such that the natural homomorphism of $A_1(M)$ into $\pi_0\varepsilon(M)$ is surjective and has a kernel isomorphic to the finite abelian group* $H^1(M;\mathbf{Z})/\mathrm{Center}(\pi_1 M)$.

The idea is to refine the Seifert fibered space construction designed by Corner and Raymond **[C-R]** in order to obtain certain criteria for a group to be embedded into the affine group. From now on, M denotes a closed n-dimensional flat Riemannian

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manifold, $\pi_1 M = \pi$ and $\operatorname{rank}(\mathfrak{z}(\pi)) = k$, where $\mathfrak{z}(\pi)$ is the center of π. We consider π as a discrete uniform subgroup of $E(n)$. Then $\operatorname{Inn}(\pi) = \pi/\mathfrak{z}(\pi)$ naturally becomes an $(n-k)$-dimensional crystallographic group.

LEMMA 1. *There exists a faithful representation of* $\operatorname{Aut}(\pi)$ *into* $\operatorname{GL}(k, \mathbf{Z}) \times A(n-k)$.

PROOF. Since $\mathfrak{z}(\pi)$ is characteristic in π, any automorphism θ of π induces an automorphism of $\mathfrak{z}(\pi)$ and $\operatorname{Inn}(\pi)$, say $\phi(\theta)$ and $\bar{\theta}$, respectively. A theorem of Bieberbach says that any automorphism of a crystallographic group is given by the conjugation by an affinity. This means $\bar{\theta} = \mu(h)$, conjugation by some $h \in A(n-k)$. Furthermore, such an h is unique because $\operatorname{Inn}(\pi)$ has trivial center. Put $\rho(\theta) = h$. Clearly ϕ and ρ are homomorphisms.

It remains to show $\phi \times \rho$ is injective. It is well known that the subgroup of automorphisms of π which induce the identity on both $\mathfrak{z}(\pi)$ and $\operatorname{Inn}(\pi)$ is isomorphic to $H^1(\operatorname{Inn}(\pi); \mathfrak{z}(\pi))$. See **[We**, 5.1.5], for example. However, $\operatorname{Inn}(\pi)$ acts trivially on $\mathfrak{z}(\pi)$ and is centerless. This implies $H^1(\operatorname{Inn}(\pi); \mathfrak{z}(\pi)) = 0$. Therefore, the injectivity of $\phi \times \rho$ has been proved.

Now we describe the injective Seifert fibered space construction in a very special case. For the general construction, see P. Conner and F. Raymond's articles, for example, **[C-R]**. See also the forthcoming paper **[L-2]** about results of restricting the coefficient space $\operatorname{MAPS}(W, \mathbf{R}^k)$.

LEMMA 2. *Let Q be a group with a faithful representation $\phi \times \rho$: $Q \to \operatorname{GL}(k, \mathbf{Z}) \times A(n-k)$. Then there is an isomorphism of $H^1(Q, T^k)$ onto the group of conjugacy classes (by $\mathbf{R}^k$) of subgroups of $A(k) \times A(n-k)$ which are extensions of $\mathbf{Z}^k$ by Q, compatible with the representation $\phi \times \rho$.*

PROOF. The action of Q on $\mathbf{Z}^k$ via ϕ naturally extends to one on $\mathbf{R}^k$. The exact sequence of Q-modules $0 \to \mathbf{Z}^k \to \mathbf{R}^k \to T^k \to 0$ induces a long exact sequence of cohomology $\cdots \to H^1(Q, \mathbf{R}^k) \to H^1(Q, T^k) \xrightarrow{\delta} H^2(Q, \mathbf{Z}^k) \to \cdots$. Given $[m] \in H^1(Q, T^k)$, choose a 1-cocycle m: $Q \to T^k$. Lift it to $\tilde{m}$: $Q \to \mathbf{R}^k$ taking care that $\tilde{m}(1) = 0$. Let $E(\tilde{m}) = \{(\phi(\alpha), \tilde{m}(\alpha) + z), \rho(\alpha)\colon\ \alpha \in Q,\ z \in \mathbf{Z}^k\} \subset A(k) \times A(n-k) \subset A(n)$. We show that $E(\tilde{m})$ is a subgroup of $A(n)$.

$$\begin{aligned}(\phi(\alpha), \tilde{m}(\alpha) + z)(\phi(\beta), \tilde{m}(\beta) + z') &= (\phi(\alpha\beta), \tilde{m}(\alpha) + z + \phi(\alpha)\tilde{m}(\beta) + \phi(\alpha)z') \\ &= (\phi(\alpha\beta), \tilde{m}(\alpha\beta) + z + \phi(\alpha)z' + \delta\tilde{m}(\alpha, \beta)).\end{aligned}$$

Note that $\delta\tilde{m}(\alpha, \beta) \in \mathbf{Z}^k$ by construction so that $z + \phi(\alpha)z' + \delta\tilde{m}(\alpha, \beta) \in \mathbf{Z}^k$. In fact, $E(\tilde{m})$ is an extension of $\mathbf{Z}^k$ by Q representing the class $\delta[m] \in H^2(Q, \mathbf{Z}^k)$.

Suppose $\tilde{m} = g + \delta\tilde{\lambda}$ for some g: $Q \to \mathbf{Z}^k$ and $\tilde{\lambda} \in \mathbf{R}^k$. Then $E(\tilde{m}) = E(\delta\tilde{\lambda})$ and $\mu(\tilde{\lambda}) =$ conjugation by $\tilde{\lambda}$ is an isomorphism of $E(\tilde{0}) = \mathbf{Z}^k \cdot Q$ to $E(\delta\tilde{\lambda})$. This proves that $E(\tilde{m})$ is unique up to conjugacy for any choice of $m \in [m]$ and the lift $\tilde{m}$. Conversely, if $E(\tilde{m})$ is conjugate to $E(\tilde{0})$ via $\mu(\tilde{\lambda})$, then $E(\tilde{m}) = E(\delta\tilde{\lambda})$, and hence $[m] = 0$ in $H^1(Q, T^k)$.

In order to show the surjectivity, let E be a subgroup of $A(k) \times A(n-k)$ which is an extension of $\mathbf{Z}^k$ by Q. Furthermore, suppose the diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}^k & \to & E & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow h & & \downarrow \phi \times \rho & & \\ 1 & \to & \mathbf{R}^k & \to & A(k) \times A(n-k) & \to & \mathrm{GL}(k, \mathbf{Z}) \times A(n-k) & \to & 1 \end{array}$$

commutes. We may assume that E is $\mathbf{Z}^k \times Q$ with multiplication $(z, \alpha)(z', \beta) = (z + \phi(\alpha)z' + f(\alpha, \beta), \alpha\beta)$ for some 2-cocycle f: $Q \times Q \to \mathbf{Z}^k$. Define $\tilde{m}$: $Q \to \mathbf{R}^k$ by $\tilde{m}(\alpha) = \mathbf{R}^k$-component of $h(0, \alpha)$. Then clearly $\tilde{m}(\alpha, \beta) = \phi(\alpha)\tilde{m}(\beta) - \tilde{m}(\alpha\beta) + \tilde{m}(\alpha) = h(0, \alpha)h(0, \beta)h(0, \alpha\beta)^{-1} \in \mathbf{Z}^k$ so that $\tilde{m}$: $Q \to \mathbf{R}^k \to T^k$ is a coboundary. This shows $h(E) = E(\tilde{m})$ for $[m] \in H^1(Q, T^k)$. Therefore, the homomorphism of $H^1(Q, T^k)$ into the group of conjugacy classes of subgroups of $A(k) \times A(n-k)$ which satisfy the conditions is surjective.

Remarks. (1) The above construction does not require $\phi \times \rho(Q)$ to be discrete. (2) If E and E' are conjugate to each other by $\tilde{\lambda} \in \mathbf{R}^k$, one can deform one to the other by $\mu(t\tilde{\lambda})$, $0 \leq t \leq 1$, in $A(k) \times A(n-k)$. (3) Not all extensions of $\mathbf{Z}^k$ by Q can be imbedded in $A(k) \times A(n-k)$. This will be clear from the next

Proposition. *An abstract kernel (G, π, φ) can be realized as a group of affine diffeomorphisms of M if and only if the kernel of φ is finite and there exists an admissible extension E of π by G realizing φ so that $1 \to C_E(\pi) \to E \to E/C_E(\pi) \to 1$ has finite order in $H^2(E/C_E(\pi); C_E(\pi))$.*

Proof. Let $1 \to \pi \to E \to G \to 1$ be an extension satisfying the conditions above. Since $0 \to \mathfrak{z}(\pi) \to C_E(\pi) \to \mathrm{kernel}(\varphi) \to 1$ is exact, $C_E(\pi)$ is a torsion free central extension of $\mathfrak{z}(\pi)$ by the finite group, kernel(φ), and hence is free abelian of rank k. (See the proof of [**L-R1**, Proposition 2].) In this proof we denote $C_E(\pi)$ by $\mathbf{Z}^k$ and $E/C_E(\pi)$ by Q. Since μ: $E \to \mathrm{Aut}\,\pi$ has kernel $C_E(\pi)$, we may consider Q as a subgroup of $\mathrm{Aut}\,\pi$. By Lemma 1, Q has a representation $Q \to \mathrm{Aut}\,\pi \to \mathrm{GL}(k, \mathbf{Z}) \times A(n-k)$. We look at the long exact sequence in cohomology $\cdots \to H^1(Q, T^k) \xrightarrow{\delta} H^2(Q, \mathbf{Z}^k) \to H^2(Q, \mathbf{R}^k) \to \cdots$ induced by the exact sequence of Q-modules $0 \to \mathbf{Z}^k \to \mathbf{R}^k \to T^k \to 0$. Let $1 \to \mathbf{Z}^k \to E \to Q \to 1$ represents $[a] \in H^2(Q, \mathbf{Z}^k)$. Since $[a]$ has finite order, it is 0 in $H^2(Q, \mathbf{R}^k)$. This implies that $[a] = \delta[m]$ for some $[m]$ in $H^1(Q, T^k)$. Then one can apply Lemma 2 to embed E into $A(n)$. Let $\tilde{m}$: $E \to A(k) \times A(n-k)$ be such an imbedding. Now, the trouble is that $\tilde{m}$ restricted to π may not be the identity. However, note that the maximal abelian subgroup $\mathbf{Z}^n$ of π maps into $\mathbf{R}^n$. Therefore, we have an isomorphism $\tilde{m}$ of a Bieberbach group π into $A(n)$ so that $\tilde{m}(\mathbf{Z}^n) \subset \mathbf{R}^n$. It is proved in [**L-R1**, Theorem 6] that such an isomorphism is, in fact, a conjugation by an $h^{-1} \in A(n)$. Thus we have obtained a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 1 & \to & \pi & \to & E & \to & G & \to & 1 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 1 & \to & \pi & \to & \mu(h)\tilde{m}(E) & \to & \mu(h)\tilde{m}(E)/\pi & \to & 1 \end{array}$$

Since $\mu(h)\tilde{m}(E) \subset A(n)$, $\mu(h)\tilde{m}(E)/\pi \subset \mathrm{Aff}(M)$ which realizes the abstract kernel $\varphi\colon G \to \mathrm{Out}\,\pi$.

COROLLARY [**L-R1**, THEOREM 3; **Z-Z**, SATZ 3.17]. *A finite abstract kernel* (G, π, φ) *can be realized as a group of affine diffeomorphisms of* M *if and only if it admits an admissible extension.*

PROOF. We have to show that if $1 \to \pi \to E \to G \to 1$ is an admissible extension, then $1 \to C_E(\pi) \to E \to E/C_E(\pi) \to 1$ has a finite order in $H^2(E/C_E(\pi);\ C_E(\pi))$. This is a nontrivial fact, but can be shown by a cohomological argument which we leave to the reader.

PROOF OF MAIN THEOREM. We have a natural homomorphism $t\colon \pi \to \pi/[\pi, \pi] = H_1(\pi; \mathbf{Z}) \to H^1(\pi, \mathbf{Z})$, where the last epimorphism is induced by the Universal Coefficient Theorem. Note that, under the homomorphism t, $\mathfrak{z}(\pi)$ injects into $H^1(\pi; \mathbf{Z})$. Since any automorphism of π induces an automorphism on $H^1(\pi; \mathbf{Z})$, $\mathrm{Aut}\,\pi$ acts on $H^1(\pi; \mathbf{Z})$. Form the semidirect product $H^1(\pi; \mathbf{Z}) \circ \mathrm{Aut}\,\pi$. We define a natural homomorphism $\nu\colon \pi \to H^1(\pi; \mathbf{Z}) \circ \mathrm{Aut}\,\pi$ by $\nu(\sigma) = (t(\sigma), \mu(\sigma))$. Noting that $\mathrm{Inn}\,\pi$ acts trivially on $H^1(\pi; \mathbf{Z})$, one can check easily that ν is a homomorphism. We claim that ν is injective. Suppose $\mu(\sigma) = 1$. Then $\sigma \in \mathfrak{z}(\pi)$. If $t(\sigma) = 0$, then $\sigma^P \in [\pi, \pi]$ for some $p > 0$. Since π is torsion free and $\mathfrak{z}(\pi) \cap [\pi, \pi] = 1$, $\nu(\sigma) = 1$ implies $\sigma = 1$. Therefore, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{z}(\pi) & \to & H^1(\pi; \mathbf{Z}) & \to & H^1(\pi; \mathbf{Z})/(\pi) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi & \to & H^1(\pi; \mathbf{Z}) \circ \mathrm{Aut}\,\pi & \to & A_1(M) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathrm{Inn}\,\pi & \to & \mathrm{Aut}\,\pi & \to & \mathrm{Out}\,\pi & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Note that the abstract kernel $\varphi\colon A_1(M) \to \mathrm{Out}\,\pi$ is a finite inflation of $\mathrm{Out}\,\pi$ itself. In order words, kernel(φ) is a finite abelian group $H^1(\pi; \mathbf{Z})/\mathfrak{z}(\pi)$. The middle row satisfies all the conditions in the proposition so that $A_1(M)$ really sits inside $\mathrm{Aff}(M)$.

REMARK. There is a certain relation between $H^1(\pi; \mathbf{Z})/\mathfrak{z}(\pi)$ and the Calabi fibration. Calabi noted that M is covered by $T^k \times N^{n-k}$, where T^k is a flat torus of dimension $k = $ rank of $\mathfrak{z}(\pi)$ and N^{n-k} is a closed flat Riemannian manifold, with an abelian covering group A. The group A acts on $T^k \times N$ diagonally, as translations on T^k and as isometries on N. See [**W**, Theorem 3.6.3] for more details. We can inflate A-action to $A \times A$-action on $T^k \times N$ so that $(\alpha, \beta)(x, y) = (\alpha x, \beta y)$ for $(\alpha, \beta) \in A \times A$ and $(x, y) \in T^k \times N$. Note that $(A, T^k \times N)$ is naturally imbedded in $(A \times A,\ T^k \times N)$ as the diagonal. Thus, $(A \times A,\ T^k \times N)$ induces an action of $A \times A/A \cong A$ on M. It is not hard to verify that $(H^1(M; \mathbf{Z})/\mathfrak{z}(\pi), M)$ is exactly the same as $(A \times A/A, M)$. Therefore the group $A_1(M)$ constructed in the theorem is the smallest subgroup of $\mathrm{Aff}(M)$ which contains $A \times A/A$ and maps onto $\mathrm{Out}\,\pi$.

COROLLARY. *Given a finite subgroup G of $\pi_0\mathcal{E}(M)$, there always exists a group G^* together with a surjective homomorphism $G^* \to G$ with a finite abelian kernel so that it can be realized as a group of affine diffeomorphisms of M. Furthermore, the finite abelian kernels are uniformly bounded by $H^1(M; \mathbf{Z})/\mathfrak{z}(\pi)$.*

REMARK. The main theorem has been stated in **[L-R1**, Theorem 5] to prove a more general form of the corollary, and was formally presented at the A.M.S. Meeting (see **[L1]**). Recently, the author has learned that a part of the corollary has been proved earlier by B. Zimmermann using the obstruction theory to the existence of group extensions. (See B. Zimmermann, *Über Gruppen von Homömorphismen Seifertscher Faserräume und Flacher Mannigfaltigkeiten*, Manuscripta Math. **30** (1980), 361–373.) Note, however, that his argument does not give any information about the kernel of the inflation. In fact, to obtain a uniform bound, and more specifically the main theorem itself, one needs not only the existence, but also a new embedding argument like the proposition.

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