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MAXIMAL INTERSECTING FAMILIES OF FINITE SETS AND n-UNIFORM HJELMSLEV PLANES

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ABSTRACT. The following theorem is proved. The collection of lines of an n-uniform projective Hjelmslev plane is maximal when considered as a collectiion of mutually intersecting sets of equal cardinality.

1. Introduction. A clique of k-sets is a collection of mutually intersecting sets of size k. We write $N(k)$ to denote the minimum cardinality of a maximal clique of k-sets. Apparently the exact value of $N(k)$ is known only for very small values of k. However, Erdös and Lovàsz [7] have obtained the asymptotic lower bound $N(k) \geqslant (8k/3) - 3$; and Füredi [**8**, p. 283] writes that he can prove $N(k) < k^{f(k)}$ where $f(k) = ck^{7/12}$.

For particular values of k, the preceding upper bound can be greatly sharpened. It is easily proved, for example, that

(1.1) a projective plane of order r is a maximal clique. Consequently

(1.2) $N(r+1) \leqslant r^2 + r + 1$ whenever r is the order of a projective plane.

In addition Füredi has proved the following two theorems (Proposition 1 and Theorem 1 in [**8**]). (Füredi informs us that (1.3) is joint work with L. Babai.)

(1.3) $N(r^2 + r) \leqslant r^4 + r^3 + r^2$ whenever r is the order of a projective plane.

(1.4) $N(2r) \leqslant 3r^2$ whenever r is the order of a projective plane.

In this paper we obtain the following common generalization of (1.2) and (1.3).

THEOREM 1.1. *If r is the order of a finite projective plane, then $N(r^n + r^{n-1}) \leqslant r^{2n} + r^{2n-1} + r^{2n-2}$ for every positive integer n.*

Füredi proves (1.3) by constructing a 2-uniform projective Hjelmslev plane over an arbitrary finite projective plane and then observing that such Hjelmslev planes are maximal cliques. Henceforth we write PH-plane for projective Hjelmslev plane. The more difficult of the two steps in the Füredi program is the PH-plane construction, a construction which has been discovered independently by Füredi [**8**] and Craig [**3**] (see also Lüneburg [**13**]). Since the class of 1-uniform PH-planes is by definition just the class of finite projective planes, conclusions (1.2) and (1.3) both follow by observing that the line set of every n-uniform PH-plane with $n = 1$ or 2 is a maximal clique. Similarly we shall obtain Theorem 1.1 as a corollary to the following result.

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THEOREM 1.2. *The line set of every (finite) n-uniform projective Hjelmslev plane is a maximal clique.*

The contribution of this paper is to prove Theorem 1.2. The other step, that of establishing the existence of n-uniform PH-planes over arbitrary projective planes, has already been completed: first by Artmann [**1**] and later by Drake [**6**] who used a different construction.

If k can be represented both as $r^m + r^{m-1}$ and as $s^p + s^{p-1}$ with $m < p$, one should apply Theorem 1.1 with $n = p$ to obtain the sharper bound. Unfortunately such double representations occur for prime powers r and s only when $m = 1$ and in the case $2^3 + 2^2 = 3^2 + 3 = 11 + 1$. In the latter case one obtains $N(12) \leqslant 133$ by using (1.2), $N(12) \leqslant 117$ by using the Füredi result (1.3), and $N(12) \leqslant 112$ by using Theorem 1.1 with $n = 3$. The real value of Theorem 1.1, of course, is that variation in n allows one to obtain a bound for $N(k)$ for new values of k.

2. Prerequisites. We refer the reader to [**5**, pp. 192–197] for background material that includes the definitions of PH-planes and NAH-planes (near affine Hjelmslev planes). We repeat here some of the material from the cited pages, however, because the conclusions of this paper will interest a number of mathematicians without previous knowledge of Hjelmslev planes. We use the designation *H-planes* to refer collectively to NAH- and PH-planes.

To every H-plane E is associated a canonical (incidence-structure) epimorphism $\phi: E \to E'$ where E' is a projective plane if E is a PH-plane and an affine plane if E is an NAH-plane. Points P and Q (lines g and h) are called *neighbors*, and one writes $P \sim Q$ ($g \sim h$), if and only if $P^\phi = Q^\phi$ ($g^\phi = h^\phi$). One writes $\nsim$ for the negation of $\sim$. *Intersecting* lines g and h satisfy $g \sim h$ if and only if $|g \cap h| > 1$. We write (P) to denote the set $\{Q: Q \sim P\}$ and (g) to denote the set $\{h: h \sim g\}$. The following result was proved by Klingenberg [**10**, Satz 3.6]. (See also the remarks on page 260 of [**12**].)

PROPOSITION 2.1. *Let the incidence structure $A = A(H, h)$ be obtained from a* PH-*plane H by removing a neighbor class (h) of lines as well as all points of H which are incident with lines of (h). Then A is an* NAH-*plane.*

To each finite H-plane E are associated three integers denoted by r, s and t. For any flag (P, g) the integer t is the number of lines h through P which satisfy $h \sim g$ (as well as the number of points Q on g that satisfy $Q \sim P$); $|(P)| = |(g)| = t^2$; $s + t$ is the number of lines incident with P; and r is the order of E'. Every line contains $s + t$ points if E is a PH-plane, s points if E is an NAH-plane. The equality $s = rt$ holds for all H-planes. The preceding properties of r, s and t were first noted (for PH-planes only) by Kleinfeld [**9**]. Accordingly we shall designate this collection of properties the *Kleinfeld Counting Lemma*.

A *nearly* 1-*uniform* PH-plane (NAH-plane) is a finite projective plane (finite affine plane). For $n > 1$ a finite H-plane E (of either type) is called *nearly n-uniform* if, for every point P, (1) E induces an incidence structure $A(P)$ on (P) which is a nearly $(n-1)$-uniform NAH-plane, (2) every line of $A(P)$ is induced by d lines of E for some fixed integer d. Proposition 1.10(11) of [5] asserts that $d = r$. A nearly n-uniform H-plane is said to be *n-uniform* if every $A(P)$ is an $(n-1)$-uniform NAH-plane with a "parallelism," but the reader will not need to understand this notion.

We now establish some conventions. All H-planes in this paper are assumed to be nearly n-uniform for some n. The symbols E_n, H_n and A_n denote a nearly n-uniform H-, PH- and NAH-plane, respectively, with E', H' and A' as the respective underlying planes. In all cases the order of the underlying plane is assumed to be r.

One writes $P(\simeq i)Q$ to mean that P and Q are joined by precisely r^i lines for $0 \leq i < n$ and $P(\simeq n)Q$ to mean that $P = Q$. One writes $P(\sim i)Q$ if $P(\simeq j)Q$ for some $j \geq i$. The negation of $P(\sim i)Q$ is denoted by $P(\nsim i)Q$. The following result is part of Proposition 1.10 of [5]; most of the proof, however, is given in the proof of Proposition 2.2 in [4] rather than in [5].

PROPOSITION 2.2. *Every nearly n-uniform* H-*plane* E_n *has the following properties.*

(1) $s = r^n$, $t = r^{n-1}$.

(2) *If P and Q are distinct points of* E_n, *then* $P(\simeq i)Q$ *for some nonnegative integer* $i < n$.

(3) *The dual of* (2) *holds for intersecting lines.*

(4) *If P is in g and* $i \geq 1$, *then* $|\{Q \in g\colon Q(\sim i)P\}| = r^{n-i}$.

(5) *The dual of* (4) *holds.*

One of the principal results of [4] (Proposition 4.6) asserts that the dual of a "strongly" n-uniform PH-plane is a strongly n-uniform PH-plane. In [14, Satz 1] Törner proves that every nearly n-uniform PH-plane is a strongly n-uniform PH-plane; Theorem 2.3 below is an immediate consequence. (An alternative proof is given in [11].)

THEOREM 2.3. *Every nearly n-uniform* PH-*plane is n-uniform, and the dual of an n-uniform* PH-*plane is an n-uniform* PH-*plane.*

Two lines g and h of A_n are said to be *quasiparallel* (and one writes $g \mid h$) if $g^\Phi \parallel h^\Phi$ in A'. Then $\mid$ is an equivalence relation which partitions the lines of A_n into $r+1$ quasiparallel classes; each such class is the disjoint union of r neighbor classes of lines, hence consists of rt^2 lines. As observed in [5, p. 202], the condition $g \mid h$ holds if and only if $|g \cap h| \neq 1$. This characterization of the quasiparallel relation makes it easy to prove the following lemma.

LEMMA 2.4. *Let g, h and P be lines and point of* E_n *such that* $g' = g \cap (P)$ *and* $h' = h \cap (P)$ *are not empty. Then* $g \sim h$ *if and only if* $g' \mid h'$ *in* $A(P)$.

3. Preliminary results.

PROPOSITION 3.1. *Let* Λ *be a quasiparallel class of* A_n, $S \subset \Lambda$, $|S| < s = r^n$. *Then there is a set C of points of* A_n *which has the following properties*: (1) $|C| = s$; (2) *each pair of points of C is joined by a line of* Λ; (3) *no point of C lies on any line of S.*

PROOF. For $n = 1$, Λ is a parallel class, and C may be taken to be the set of points of any line in $\Lambda \setminus S$. Assume $n > 1$, and let $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ be the r line neighborhoods contained in Λ. If S_j denotes $S \cap \Lambda_j$ for each j, then $|S_i| < s/r = t$ for some i. We intend to obtain C from the set of points that are incident with lines of Λ_i. Let h be a line in Λ_i; $P_1, P_2, \ldots, P_r$ be r mutually nonneighbor points on h. For arbitrary fixed j, let $\Lambda' = \{g' \colon g' = g \cap (P_j)$ for some g in $\Lambda_i\}$, $S' = \{g' \colon g' = g \cap (P_j)$ for some g in $S_i\}$. By Lemma 2.4, Λ' is a quasiparallel class of lines in the nearly $(n-1)$-uniform NAH-plane $A(P_j)$: and S' is a subset of fewer than $t = r^{n-1}$ lines of Λ'. By the induction assumption there is a set $C_j \subset (P_j)$ such that (1) $|C_j| = r^{n-1}$; (2) each pair of points of C_j is joined by a line of Λ; (3) no point of C_j lies on any line of S. We take C to be the union of the C_j.

PROPOSITION 3.2. *Let g be any line of H_n, $N \subset (g)$, $|N| < t$. Then there is a set D of points of H_n with the properties*: (1) $|D| = s + t$; (2) *each pair of points of D is joined by a line of* (g); (3) *no point of D lies on any line of N.*

PROOF. Let $P_0, P_1, \ldots, P_r$ be $r + 1$ mutually nonneighbor points on g. For fixed $j \geq 0$, apply Lemma 2.4 to see that the lines of N induce a subset N' of a quasiparallel class of lines in $A(P_j)$. Applying Proposition 3.1 (with $n - 1$ instead of n), we obtain a set D_j of points of (P_j) such that (1) $|D_j| = t$; (2) each pair of points of D_j is joined by a line of (g); (3) no point of D_j lies on any line N. We now take D to be the union of the D_j.

PROPOSITION 3.3. *Let S be a set of at most $s + t$ mutually intersecting lines of A_n whose union contains every point of A_n. Then all lines of S pass through a common point.*

PROOF. The assertion is easily verified for $n = 1$, so assume $n > 1$. Let $g_1', g_2', \ldots, g_d'$ be the distinct images in A' of the lines of S. Since the g_i' intersect in A', $d \leq r + 1$. Then the g_i' pass through a common point P', and hence the lines of S all contain points from a common neighborhood (P). The number of points of A_n not in (P) is $t^2(r^2 - 1) = s^2 - t^2$, and each line of S contains $s - t$ points outside (P). Then every point outside (P) must lie on a single line of S, so every pair of lines of S must intersect in (P). Let g be any line of S. Applying Proposition 2.2(5) with $i = n - 1$, one sees that there are $r - 1$ other lines h which satisfy $h \cap (P) = g \cap (P)$. Take Q to be any point of $h \setminus (P)$, and let k be a line of S which contains Q. Then k and g intersect in $g \cap (P) = h \cap (P)$. Then $k \cap h$ contains nonneighbor points, so $h = k$ is in S. It follows that the set $S^* = \{g \cap (P) \colon g \in S\}$ has cardinality at most $(s + t)/r = r^{n-1} + r^{n-2}$. Applying the induction assumption to $A(P)$, we see that all lines of S^* (and therefore all lines of S) meet in a common point.

4. Proofs of the main results. Thanks to Theorem 2.3, it is immaterial whether we prove Theorem 1.2 or its dual. Then let S be a set of $s + t$ or fewer lines of H_n whose union contains every point of H_n. To complete the proof of Theorem 1.2 it suffices to prove the existence of a point P which lies on all lines of S. We intend to apply Proposition 3.3. To do so, we must remove a neighbor class (h) of lines from H_n to obtain a nearly n-uniform NAH-plane A_n (see Proposition 2.1). This must be done so that the intersections of lines of S lie in A_n.

For any g in S let N denote $S \cap (g)$. Assume $|N| < t$, and apply Proposition 3.2 to obtain a set D of $s + t$ points. Conditions (2) and (3) of Proposition 3.2 guarantee that the points of D lie on at least $s + t$ lines of $S \setminus (g)$. Since g is in S, we have produced the contradiction $|S| > s + t$. Then $|S \cap (g)|$ must be at least t for every g in S, so S contains lines from at most $(s + t)/t = r + 1$ distinct line neighborhoods of H_n. Consider the image S^ϕ of S in H', and apply the dual of (1.1): one sees that S^ϕ is the set of all $r + 1$ lines incident with some point Q' of H'. Then S contains exactly $t(r + 1) = s + t$ lines. Let Q be a point of H_n with $Q^\phi = Q'$. The number of flags (R, g) with g in S and $R \not\sim Q$ is $(s + t)s = t^2(r^2 + r)$; i.e., is just the number of points R of H_n with $R \not\sim Q$. Then every point $R \not\sim Q$ lies on a unique line of S, so all intersections of pairs of lines of S lie in (Q). Let h be any line having an empty intersection with (Q). Applying Proposition 3.3 to $A_n = A(H_n, h)$ completes the proof of Theorem 1.2.

To prove Theorem 1.1, let r be the order of a projective plane, n be a positive integer. Then there exists an n-uniform PH-plane H_n whose associated plane H' is of order r: this assertion is Corollary 8 of **[6]**; it is also the main theorem of **[1]** if one uses the Bacon result **[2]** that finite PH-planes of level n are n-uniform. Now Theorem 1.1 follows from Theorem 1.2 in view of Proposition 2.2(1) and the Kleinfeld Counting Lemma.

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REFERENCES

1. B. Artmann, *Existenz und projektive Limiten von Hjelmslev-Ebenen n-ter Stufe*, Atti del Convegno di Geometria Combinatoria e sue Applicazioni, Perugia, 1971, pp. 27–41.

2. P. Y. Bacon, *Strongly n-uniform and level n Hjelmslev planes*, Math. Z. **127** (1972), 1–9.

3. R. T. Craig, *Extensions of finite projective planes*. I. *Uniform Hjelmslev planes*, Canad. J. Math. **16** (1964), 261–266.

4. D. A. Drake, *On n-uniform Hjelmslev planes*, J. Combin. Theory **9** (1970), 267–288.

5. ______, *Existence of parallelisms and projective extensions for strongly n-uniform near affine Hjelmslev planes*, Geom. Dedicata **3** (1974), 191–214.

6. ______, *Constructions of Hjelmslev planes*, J. Geometry **10** (1977), 179–193.

7. P. Erdös and L. Lovàsz, *Problems and results on 3-chromatic hypergraphs and some related questions*, Proc. Colloq. Math. Soc. J. Bolyai, no. 10, North-Holland, Amsterdam, 1974, pp. 609–627.

8. Z. Füredi, *On maximal intersecting families of finite sets*, J. Combin. Theory Ser. A **28** (1980), 282–289.

9. E. Kleinfeld, *Finite Hjelmslev planes*, Illinois J. Math. **3** (1959), 403–407.

10. W. Klingenberg, *Projektive und affine Ebenen mit Nachbarelementen*, Math. Z. **60** (1954), 384–406.

11. B. V. Limaye and S. S. Sane, *On partial designs and m-uniform projective Hjelmslev planes*, J. Combin. Inform. System Sci. **3** (1978), 223–237.

12. H. Lüneburg, *Affine Hjelmslev-Ebenen mit transitiver Translationsgruppe*, Math. Z. **79** (1962), 260–288.

13. ______, *Kombinatorik*, Birkhäuser Verlag, Basel, 1971.

14. G. Törner, *n-uniforme projektive Hjelmslev-Ebenen sind stark n-uniform*, Geom. Dedicata **6** (1977), 291–295.

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