ON HEMINEARNESS SPACES

MARYAM SHAYEGAN HASTINGS¹

ABSTRACT. The concept of nearness was introduced by H. Herrlich [4'], where nearness refers to collections of sets. Nearness spaces have proven to be an extremely useful generalization of a wide class of topological structures.

A category which contains *NEAR* as a subcategory is the category *H-NEAR*, of hemineariness spaces. It arises naturally while studying products of nearness spaces. A nice embedding theorem further motivated us to make a detailed study of this category. H. Herrlich [4'] has described a process by which one can construct completions for nearness spaces. *H-NEAR* is the only supercategory of *NEAR* known to us where this process of completion can be imitated satisfactorily.

In §I we define hemineariness, give some examples, describe initial, final, product, coproduct, subobject, quotient object, and completion for hemineariness spaces.

In §II we introduce the concept of preclosure, semiclosure, and hemiclosure. We show that any hemineariness space can be embedded as a subspace of a closure hemineariness space.

I. Let X be a set and $\xi \subset P^2X$ and consider the following axioms:

(N1) If $\mathfrak{U} \in \xi$ and $\mathfrak{A}$ corefines $\mathfrak{U}$ (i.e., for each $A \in \mathfrak{A}$ there exists a $B \in \mathfrak{U}$ such that $B \subset A$) then $\mathfrak{A} \in \xi$.

(N2) If $\bigcap \mathfrak{A} \neq \emptyset$, then $\mathfrak{A} \in \xi$.

(N3) $\emptyset \neq \xi \neq P^2X$.

(N4) If $\mathfrak{A} \vee \mathfrak{U} = \{A \cup B \mid A \in \mathfrak{A}, B \in \mathfrak{U}\} \in \xi$, then $\mathfrak{A} \in \xi$ or $\mathfrak{U} \in \xi$.

(N5) If $\text{Cl}_\xi \mathfrak{A} \in \xi$, then $\mathfrak{A} \in \xi$. ($\text{Cl}_\xi A = \{x \in X \mid \{\{x\}, A\} \in \xi\}$ and $\text{Cl}_\xi \mathfrak{A} = \{\text{Cl}_\xi A \mid A \in \mathfrak{A}\}$.)

1. **DEFINITION [5].** (X, ξ) is called a *prenearness* [*semineariness*, *nearness*] space if and only if ξ satisfies (N1)–(N3) [(N1)–(N4), (N1)–(N5)]. These spaces were introduced by H. Herrlich [5].

2. **DEFINITION [5].** If (X, ξ) and (Y, η) are prenearness spaces, then a function $f: (X, \xi) \rightarrow (Y, \eta)$ is called a *nearness-preserving* map if and only if $\mathfrak{A} \in \xi$ implies that $f(\mathfrak{A}) \in \eta$.

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We consider those spaces where ξ satisfies (N1)–(N3) and (N5). We call them *hemineariness* spaces; they were first introduced by Hunsaker and Sharma [9].

We will give some examples of hemineariness spaces, and describe the relation between the category of hemineariness spaces and the categories of prenearness spaces and nearness spaces. Initial, final, product, coproduct, subobjects, and quotient objects are described in the category of hemineariness spaces. We also describe completions for hemineariness spaces.

3. EXAMPLES. (a) Let $\mathbf{R}$ denote the reals and for each $A \subseteq \mathbf{R}$ define $\text{Cl } A =$ convex hull of A . Let $\xi = \{\mathcal{A} \subset P\mathbf{R} \mid \bigcap \text{Cl } \mathcal{A} \neq \emptyset\}$. Then ξ is a hemineariness structure on $\mathbf{R}$ which is not a nearness structure. Note that if A and B are subsets of $\mathbf{R}$, $\text{Cl}(A \cup B) \neq \text{Cl } A \cup \text{Cl } B$ in general.

(b) Let $(\mathbf{R}, \xi)$ be defined as follows: Let $(\mathbf{R}, \text{Cl})$ denote the reals with the usual topology and $\xi = \{\mathcal{A} \subset P\mathbf{R} \mid \bigcap \text{Cl } \mathcal{A} \neq \emptyset\}$, then $(\mathbf{R}, \xi)$ is a nearness space. Let $\eta = \{\mathcal{A} \subset P(\mathbf{R} \times \mathbf{R}) \mid Pr_i \mathcal{A} \in \xi \text{ for } i = 1, 2\}$, where $Pr_i: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is the canonical projection for $i = 1, 2$. Then η is a hemineariness structure on $\mathbf{R} \times \mathbf{R}$ which is not a nearness structure. In particular, if A and B are subsets of $\mathbf{R} \times \mathbf{R}$, $\text{Cl}(A \cup B) \neq \text{Cl } A \cup \text{Cl } B$ in general.

The category of prenearness [semineariness, hemineariness, nearness] spaces and nearness preserving maps is called *P-NEAR* [*S-NEAR*, *H-NEAR*, *NEAR*].

4. THEOREM. *NEAR is a bicoreflective subcategory of H-NEAR. For any hemineariness space (X, ξ) , the map $1_X: (X, \xi_s) \rightarrow (X, \xi)$ is the NEAR coreflection of (X, ξ) where ξ_s is defined as in [5] as follows: $\mathcal{A} \notin \xi_s$ if and only if there exists $\mathcal{A}_1 \cdots \mathcal{A}_n \notin \xi$ such that $\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n$ corefines $\mathcal{A}$.*

5. THEOREM. *H-NEAR is a bireflective subcategory of P-NEAR. For any P-Near space (X, ξ) , the map $1_X: (X, \xi) \rightarrow (X, \xi_n)$ is the H-NEAR bireflection of (X, ξ) , where ξ_n is defined as in [5] as follows:*

- (1) $\text{Cl}^0 A = A$.
- (2) $\text{Cl}^{\alpha+1} A = \{x \in X \mid \{\text{Cl}^\alpha A, \text{Cl}^\alpha \{x\}\} \in \xi\}$.
- (3) $\text{Cl}^\beta A = \bigcup \{\text{Cl}^\alpha A \mid \alpha < \beta\}$ if β is a limit ordinal. $\text{Cl } A = \bigcup \{\text{Cl}^\alpha A \mid \alpha \text{ is an ordinal}\}$. $\xi_n = \{\mathcal{A} \subset PX \mid \{\text{Cl } A \mid A \in \mathcal{A}\} \in \xi\}$.

PROOF. Observe that if α is an ordinal greater than or equal to the cardinal of X , then $\text{Cl } A = \text{Cl}^\alpha A$ for each $A \subset X$.

6. THEOREM. *For any set X , the set of all hemineariness structures on X is a complete lattice with respect to inclusion. Especially*

- (1) *The discrete nearness* $= \{\mathcal{A} \subset PX \mid \bigcap \mathcal{A} \neq \emptyset\}$ *is the smallest element.*
- (2) *The indiscrete nearness* $= \{\mathcal{A} \subset PX \mid \emptyset \notin \mathcal{A}\}$ *is the largest element.*
- (3) *If Ω is a set of hemineariness structures on X , $\inf \Omega = \bigcap \Omega$ and $\sup \Omega = (\bigcup \Omega)_n$.*

7. PROPOSITION. *If X is a set, $(X_i, \xi_i)_{i \in I}$ a family of hemineariness spaces, then*

(1) *If $(f_i: X \rightarrow X_i)_{i \in I}$ is a family of maps, $\xi = \bigcap \{f_i^{-1}(\xi_i) \mid i \in I\}$ is a hemineariness structure on X initial with respect to $(X, f_i, (X_i, \xi_i))_{i \in I}$ where $f_i^{-1}(\xi_i) = \{\mathcal{A} \subset PX \mid f_i \mathcal{A} \in \xi_i\}$ for each $i \in I$.*

(2) *If $(f_i: X_i \rightarrow X)_{i \in I}$ is a family of maps,*

$$\xi = (\{\mathcal{A} \subset PX \mid \bigcap \mathcal{A} \neq \emptyset\} \cup \bigcup \{f_i(\xi_i) \mid i \in I\})_n$$

is a hemineariness structure on X final with respect to $((X_i, \xi_i), f_i, X)_{i \in I}$ where $f_i(\xi_i) = \{\mathfrak{A} \subset PX \mid f_i^{-1}\mathfrak{A} \in \xi_i\}$ for each $i \in I$.

8. COROLLARY. $H\text{-NEAR}$ is a properly fibred topological category.

9. REMARK. For definitions of subspace, quotient space, product, coproduct see Hastings [4].

10. PROPOSITION. Let (X, ξ) and (Y, η) be hemineariness spaces, then

(1) (X, ξ) is a subspace of (Y, η) if and only if there exists an injection $f: (X, \xi) \rightarrow (Y, \eta)$ such that $\mathfrak{A} \in \xi$ if and only if $f(\mathfrak{A}) \in \eta$.

(2) (Y, η) is a quotient space of (X, ξ) if and only if there exists a surjection $f: (X, \xi) \rightarrow (Y, \eta)$ such that $\mathfrak{A} \in \eta$ if and only if $f^{-1}\mathfrak{A} \in \xi$.

11. PROPOSITION. Let $(X_i, \xi_i)_{i \in I}$ be a family of hemineariness spaces then

(1) $(\prod_{i \in I} X_i, \xi)$ is the product of this family in $H\text{-NEAR}$ if and only if $\xi = \{\mathfrak{A} \subset P(\prod_{i \in I} X_i) \mid Pr_i \mathfrak{A} \in \xi_i \text{ for each } i \in I\}$.

(2) $(\prod_{i \in I} X_i, \xi)$ is the coproduct of this family in $H\text{-NEAR}$ if and only if $\xi = \{\mathfrak{A} \subset P(\prod_{i \in I} X_i) \mid inj_i^{-1}\mathfrak{A} \in \xi_i \text{ for some } i \in I\}$.

PROOF. (1) ξ is initial with respect to $(\prod X_i, pr_i, (X_i, \xi_i)_{i \in I})$ if and only if

$$\xi = \cap \{Pr_i^{-1}(\xi_i) \mid i \in I\} = \left\{ \mathfrak{A} \subset P\left(\prod_{i \in I} X_i\right) \mid Pr_i \mathfrak{A} \in \xi_i \text{ for each } i \in I \right\}.$$

(2) ξ is final with respect to $((X_i, \xi_i), inj_i, \prod X_i)_{i \in I}$ if and only if

$$\begin{aligned} \xi &= \left(\cup \{inj_i(\xi_i) \mid i \in I\} \cup \left\{ \mathfrak{L} \subset P\left(\prod_{i \in I} X_i\right) \mid \cap \mathfrak{L} \neq \emptyset \right\} \right)_n \\ &= \left(\left\{ \mathfrak{A} \subset P\left(\prod_{i \in I} X_i\right) \mid inj_i^{-1}\mathfrak{A} \in \xi_i \text{ for some } i \in I \right\} \right)_n. \end{aligned}$$

Let $\beta = \{\mathfrak{A} \subset P(\prod_{i \in I} X_i) \mid inj_i^{-1}\mathfrak{A} \in \xi_i \text{ for some } i \in I\}$. We claim that $\xi = \beta$. Suppose $Cl_\beta \mathfrak{A} \in \beta$. Then for some $i \in I$, $inj_i^{-1}(Cl_\beta \mathfrak{A}) \in \xi_i$. $inj_j(X_i) \cap inj_k(X_k) = \emptyset$ for each pair $(j, k), j \neq k$ implies that $inj_i^{-1}(Cl_\xi \mathfrak{A}) = Cl_{\xi_i}(inj_i^{-1}\mathfrak{A})$. Therefore, $inj_i^{-1}\mathfrak{A} \in \xi_i$ which implies that $\mathfrak{A} \in \beta$.

12. PROPOSITION. Suppose (X, ξ) and (Y, η) are hemineariness [prenearness] spaces and $f: (X, \xi) \rightarrow (Y, \eta)$ is an embedding. Then $f: (X, \xi_s) \rightarrow (Y, \eta_s)$ is an embedding of the nearness [semineariness] space (X, ξ_s) into (Y, η_s) .

13. PROPOSITION. Let $(X_i, \beta_i)_{i \in I}$ be a family of hemineariness spaces and let $\xi_i = (\beta_i)_s$ for each $i \in I$. Let $(\prod_{i \in I} X_i, \xi)$ be the product of $(X_i, \xi_i)_{i \in I}$ in $NEAR$ and $(\prod_{i \in I} X_i, \beta)$ the product of $(X_i, \beta_i)_{i \in I}$ in $H\text{-NEAR}$. Then $\beta_s = \xi$.

PROOF. $(\prod_{i \in I}, \xi)$ is the product of $(X_i, \xi_i)_{i \in I}$ in $NEAR$ means $\xi = \eta_s$ where $\eta = \{\mathfrak{A} \subset P(\prod_{i \in I} X_i) \mid Pr_i \mathfrak{A} \in \xi_i \text{ for each } i \in I\}$. $(\prod_{i \in I} X_i, \beta)$ is the product of $(X_i, \beta_i)_{i \in I}$ in $H\text{-NEAR}$ means $\beta = \{\mathfrak{A} \subset P(\prod_{i \in I} X_i) \mid Pr_i \mathfrak{A} \in \beta_i \text{ for each } i \in I\}$. If $\mathfrak{A} \notin \xi$, then there exists $\mathfrak{A}_1, \dots, \mathfrak{A}_n \notin \eta$ such that $\mathfrak{A}_1 \vee \dots \vee \mathfrak{A}_n$ corefines $\mathfrak{A}$ which implies that for each $1 \leq i \leq n$ there exists j_i and $\mathcal{L}_1^{j_i}, \dots, \mathcal{L}_{m_{j_i}}^{j_i} \notin \beta_{j_i}$ such that $\mathcal{L}_1^{j_i} \vee \dots \vee \mathcal{L}_{m_{j_i}}^{j_i}$ corefines $Pr_{j_i} \mathfrak{A}_i$. Therefore, $Pr_{j_i}^{-1} \mathcal{L}_q^{j_i} \notin \beta$ for each $1 \leq i \leq n$ and for each $1 \leq q \leq m_{j_i}$ and $\bigvee_{i=1}^n (\bigvee_{q=1}^{m_{j_i}} \mathcal{L}_q^{j_i}) < \mathfrak{A}$.

14. REMARK. H. Herrlich [5] developed a method by which one can construct completions for nearness spaces. Here we imitate the terminology and the construction in order to build completions for hemineariness spaces. $H\text{-NEAR}$ is the only category containing $NEAR$ known to us where the process of completion can be imitated satisfactorily.

15. DEFINITION. Let (X, ξ) be a hemineariness space. The maximal elements of the set $\xi - \{\emptyset\}$, ordered by inclusion are called ξ -clusters. A ξ -cluster $\mathfrak{A}$ is called a *point ξ -cluster* if and only if there exists $x \in X$ with $\{x\} \in \mathfrak{A}$.

16. Note. If (X, ξ) is a nearness space then a ξ -cluster is always a grill in a sense of Chouquet [3], i.e., $A \cup B \in \mathfrak{A}$ if and only if $A \in \mathfrak{A}$ or $B \in \mathfrak{A}$. But ξ -cluster in a hemineariness space generally fails to be a grill.

EXAMPLE. Let $(\mathbb{R}, \xi)$ be a hemineariness space defined as in Example 3(a) and let $x \in \mathbb{R}$. Then $e(x) = \{A \subseteq \mathbb{R} \mid x \in \text{Cl } A\}$ is a ξ -cluster. Let $a < x < b$ and $A = (-\infty, a)$, $B = (b, \infty)$. Then $A \cup B \in e(x)$, but $A \notin e(x)$ and $B \notin e(x)$.

17. PROPOSITION. If (X, ξ) is a hemineariness space and if $x \in X$, then $e(x) = \{A \subset X \mid x \in \text{Cl}_\xi A\}$ is a point ξ -cluster. Conversely, any point ξ -cluster is of this form.

18. DEFINITION. A hemineariness space is called complete if and only if every ξ -cluster is a point ξ -cluster.

19. DEFINITION. A prenearness space (X, ξ) is called a $T1$ -prenearness space if and only if for any pair of elements (x, y) of X , if $\{(x), (y)\} \in \xi$ then $x = y$.

20. THEOREM. Let (X, ξ) be a hemineariness space, let X^* be the set of all ξ -clusters, let ξ^* be the set of all $\Omega \subset PX^*$ with $(\bigcup \{\bigcap \omega \mid \omega \in \Omega\}) \in \xi$ and let $f: X \rightarrow X^*$ be the function which maps each $x \in X$ to the point ξ -cluster $e(x)$. Then the following hold:

(1) (X^*, ξ^*) is a complete $T1$ -hemineariness space.

(2) $f: (X, \xi) \rightarrow (X^*, \xi^*)$ is a nearness preserving map.

Moreover the following hold:

(a) $\mathfrak{A} \in \xi$ if and only if $f\mathfrak{A} \in \xi^*$.

(b) $\text{Cl}_{\xi^*} fX = X^*$.

(3) f is injective if and only if (X, ξ) is a $T1$ -hemineariness space.

(4) f is surjective if and only if (X, ξ) is complete.

21. REMARK. The above construction $f: (X, \xi) \rightarrow (X^*, \xi^*)$ is called the completion of (X, ξ) .

The proof of Theorem 20 is similar to that of Theorem 5.7 of [4'] and will therefore be omitted.

Note. To show that $f: X \rightarrow X^*$ is nearness preserving, we need axiom (N5). This process of completion cannot be imitated for $P\text{-NEAR}$ or $S\text{-NEAR}$.

PROOF OF NOTE. If $\mathfrak{A} \in \xi$ and $B \in \bigcup \{\bigcap \omega \mid \omega \in f(\mathfrak{A})\}$, then $B \in \bigcap fA$ for some $A \in \mathfrak{A}$ which implies that $A \subset \text{Cl}_\xi B$. Therefore by (N1) $\{\text{Cl}_\xi(\bigcup \{\bigcap \omega \mid \omega \in f(\mathfrak{A})\})\} \in \xi$ and by (N5) $\bigcup \{\bigcap \omega \mid \omega \in f(\mathfrak{A})\} \in \xi$, i.e., $f(\mathfrak{A}) \in \xi^*$.

II. Closure hemineariness spaces. We will now introduce the concept of preclosure, semiclosure, and hemiclosure, and define the category $C\text{-H-NEAR}$ of closure hemineariness spaces. We prove that $C\text{-H-NEAR}$ is bicoreflective in $H\text{-NEAR}$ and is isomorphic to the category of hemiclosure spaces and continuous functions.

In the 1973 Categorical Topology Conference at Southern Illinois University in Carbondale, H. Herrlich [7] posed the following question: how far removed is *NEAR* from *TOP*? The analogous question in the category of hemineariness spaces is: how far removed is *C-H-NEAR* from *H-NEAR*? We will show that any hemineariness space is a subspace of some closure hemineariness space. Our method can also be used to obtain analogous results in *P-NEAR*. For the case of *S-NEAR* we proved [4] that any semineariness space can be embedded as a subspace of a product of grill-determined spaces.

22. DEFINITIONS. Let X be a set. Cl an operator on X and consider the following axioms. For any pair (x, y) of elements of X and for any pair (A, B) of subsets of X :

(0) If $x \in \text{Cl}\{y\}$, then $y \in \text{Cl}\{x\}$.

(1) $\text{Cl} \emptyset = \emptyset$.

(2) $A \subset \text{Cl} A$.

(3) If $A \subset B$, then $\text{Cl} A \subset \text{Cl} B$.

(4) $\text{Cl}(A \cup B) = \text{Cl} A \cup \text{Cl} B$.

(5) $\text{Cl Cl} A = \text{Cl} A$.

(X, Cl) is called a *preclosure* [*semiclosure*, *hemiclosure*, *topological*] space if and only if Cl satisfies axioms (0)–(3) [(0)–(4), (0)–(3) and (5), (0)–(5)].

If (X, Cl_X) and (Y, Cl_Y) are preclosure spaces, then a function $f: (X, \text{Cl}_X) \rightarrow (Y, \text{Cl}_Y)$ is called continuous if and only if $f(\text{Cl}_X A) \subset \text{Cl}_Y fA$ for each $A \subset X$.

The category of preclosure [*semiclosure*, *hemiclosure*, *topological*] spaces and continuous functions is called *P-CLOS* [*S-CLOS*, *H-CLOS*, *TOP*]. Throughout this paper we require that topological spaces satisfy the symmetry axiom (0) above.

22'. REMARK. *S-CLOS* has been studied by E. Čech [2].

23. DEFINITION. A hemineariness space (X, ξ) is called a *closure hemineariness* space if and only if it satisfies the following condition:

(c) if $\mathfrak{A} \in \xi$ then $\bigcap \text{Cl}_\xi \mathfrak{A} \neq \emptyset$.

Note that this condition is equivalent to

(c') $\mathfrak{A} \in \xi$ if and only if $\bigcap \text{Cl}_\xi \mathfrak{A} \neq \emptyset$.

24. THEOREM. The full subcategory *C-H-NEAR* of *H-NEAR* whose objects are closure hemineariness spaces is:

(a) *bicoreflective in H-NEAR*. $1_X: (X, \xi_i) \rightarrow (X, \xi)$ is the *C-H-NEAR* coreflection, where $\xi_i = \{\mathfrak{A} \subset P(X) \mid \bigcap \text{Cl}_{\xi_i} \mathfrak{A} \neq \emptyset\}$.

(b) *isomorphic to the category H-CLOS of hemiclosure spaces and continuous functions*. $F: \text{C-H-NEAR} \rightarrow \text{H-CLOS}$ is an isomorphism, where $F(X, \xi) = (X, \text{Cl}_\xi)$.

25. PROPOSITION. If $(X_i, \xi_i)_{i \in I}$ is a family of hemineariness spaces, then $(X_i, \xi_i)_{i \in I}$ is a family of closure hemineariness spaces if and only if $(\prod_{i \in I} X_i, \xi)$, the product of this family in *H-NEAR* is a closure hemineariness space.

PROOF. Let $\mathfrak{A} \in \xi$, then for each $i \in I$ there exists $x_i \in \bigcap \text{Cl}_{\xi_i} \text{Pr}_i \mathfrak{A}$. Let $x \in \prod_{i \in I} X_i$ such that $\text{Pr}_i x = x_i$ for each $i \in I$, then $x \in \bigcap \text{Cl}_\xi \mathfrak{A} \neq \emptyset$. Suppose there exists $i \in I$ such that (X_i, ξ_i) is not a closure hemineariness space, i.e., there exists

$\mathfrak{A} \in \xi$, such that $\cap \text{Cl}_{\xi} \mathfrak{A} = \emptyset$. For each $A \in \mathfrak{A}$ let $B_A \subset \prod_{i \in I} X_i$ such that $Pr_i B_A = A$ and $Pr_j B_A = X_j$ for $j \neq i$; $i, j \in I$. Let $\mathcal{L} = \{B_A \mid A \in \mathfrak{A}\}$, then $\mathcal{L} \in \xi$ and $\cap \text{Cl}_{\xi} \mathcal{L} = \emptyset$. Therefore, $(\prod_{i \in I} X_i, \xi)$ is not a closure hemineariness space.

26. THEOREM. Any hemineariness space can be embedded as a subspace of a closure hemineariness space.

PROOF. Let (X, ξ) be a hemineariness space. Let $\alpha = \{\mathfrak{A} \in \xi \mid \cap \text{Cl}_{\xi} \mathfrak{A} \neq \emptyset, \mathfrak{A} \text{ is a stack (i.e., if } A \in \mathfrak{A} \text{ and } A \subset B \text{ then } B \in \mathfrak{A}) \text{ and } \text{Cl}_{\xi} A \in \mathfrak{A} \text{ implies that } A \in \mathfrak{A}\}$. For each $\mathfrak{A} \in \alpha$, adjoin $x_{\mathfrak{A}}$ to X , i.e., let $X' = X \cup \{x_{\mathfrak{A}} \mid \mathfrak{A} \in \alpha\}$. For each $A \subset X'$ define $\text{Cl } A = A \cup \text{Cl}_{\xi}(A \cap X) \cup \{x_{\mathfrak{A}} \mid \mathfrak{A} \in \alpha \text{ and } A \cap X \in \mathfrak{A}\}$. We claim that (X', Cl) is a hemiclosure space. The only nontrivial fact in establishing this claim is that $\text{Cl Cl } A \subset \text{Cl } A$ for any $A \subset X'$.

Let $x \in \text{Cl Cl } A = \text{Cl } A \cup \text{Cl}_{\xi}((\text{Cl } A) \cap X) \cup \{x_{\mathfrak{A}} \mid \mathfrak{A} \in \alpha \text{ and } (\text{Cl } A) \cap X \in \mathfrak{A}\}$. Since $(\text{Cl } A) \cap X = \text{Cl}_{\xi}(A \cap X)$ one of the following must hold:

(1) $x \in \text{Cl } A$.

(2) $x \in \text{Cl}_{\xi}(\text{Cl}_{\xi}(A \cap X)) = \text{Cl}_{\xi}(A \cap X) \subset \text{Cl } A$, i.e., $x \in \text{Cl } A$.

(3) $x = x_{\mathfrak{A}}$, where $\mathfrak{A} \in \alpha$ and $\text{Cl}_{\xi}(A \cap X) \in \mathfrak{A}$ which implies that $A \cap X \in \mathfrak{A}$, i.e., $x \in \{x_{\mathfrak{A}} \mid \mathfrak{A} \in \alpha \text{ and } A \cap X \in \mathfrak{A}\} \subset \text{Cl } A$, i.e., $x \in \text{Cl } A$. This completes the proof of $\text{Cl Cl } A \subset \text{Cl } A$ for any $A \subset X'$. Define $\xi' = \{\mathfrak{A} \subset P(X') \mid \cap \text{Cl } \mathfrak{A} \neq \emptyset\}$, then $\text{Cl}_{\xi'} A = \text{Cl } A$ for each $A \subset X'$ and (X', ξ') is a hemineariness space. Now, $\mathfrak{A} \in \xi'$ if and only if $\cap \text{Cl } \mathfrak{A} = \cap \text{Cl}_{\xi'} \mathfrak{A} \neq \emptyset$ which implies that (X', ξ') is a closure hemineariness space. We claim that the inclusion map $i: (X, \xi) \rightarrow (X', \xi')$ is an embedding. In establishing this claim we will only give the details in the proof of the case where $\mathfrak{A} \in \xi$ and $\cap \text{Cl}_{\xi} \mathfrak{A} = \emptyset$. In this case let $\mathcal{L} = \{B \subset X \mid \text{there exists } A \in \mathfrak{A} \text{ such that } \text{Cl}_{\xi} A \subset \text{Cl}_{\xi} B\}$. Then $\mathfrak{A} \subset \mathcal{L}$ and $\mathcal{L} \in \alpha$ which implies that $x_{\mathcal{L}} \in \cap \text{Cl}_{\xi'} \mathcal{L} \subset \text{Cl}_{\xi'} \mathfrak{A} \neq \emptyset$ and so $i(\mathfrak{A}) = \mathfrak{A} \in \xi'$.

27. REMARK. The construction used in this theorem results in a dense embedding, i.e., $\text{Cl}_{\xi'} X = X'$.

30. COROLLARY. The epireflective hull of C-H-NEAR in H-NEAR is H-NEAR.

31. COROLLARY. Any nearness space can be embedded as a subspace of a nearness space (X', β_s) where (X', β) is a closure hemineariness space.

We now bring together the results of §§I and II.

32. PROPOSITION. Each closure hemineariness space is complete.

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DEPARTMENT OF MATHEMATICS, WILLIAM PATERSON COLLEGE, WAYNE, NEW JERSEY 07470

Current address: Department of Mathematics, Marymount College, Tarrytown, New York 10591