INEQUALITIES OF GAUSS-BONNET TYPE FOR A CONVEX DOMAIN

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ABSTRACT. Let N be a compact convex n -dimensional Riemannian manifold with a boundary ∂N having normal curvatures $\geq \kappa > 0$. Suppose the sectional curvature $> -\kappa^2$ in N . Let H be the integral mean curvature of ∂N , V be the volume of N , k_{sc} be the scalar curvature and $\bar{k}_R(p)$, $p \in N$, be the maximum Ricci curvature at p . Then

$$H \geq \frac{n-2}{2} \kappa^2 V - \frac{1}{2(n-1)} \int_N k_{sc} dV, \quad H \geq (n-2) \kappa^2 V - \frac{1}{n-1} \int_N \bar{k}_R dV.$$

Let N_- denote N with nonpositive sectional curvature. Let G be the integral Gauss curvature of ∂N_- . Then $G \geq -\kappa^{n-2} \int_{N_-} \bar{k}_R dV$. These three estimates are sharp. For a ball in 3-dimensional hyperbolic space, the ratio of the right-hand part of each estimate to its left-hand part (i.e. $V(\kappa^2 + 3)/2H$, $V(\kappa^2 + 1)/H$ and $2\kappa V/G$ respectively) approaches 1 as the radius $\rightarrow \infty$. The same ratios for the estimates

$$H \geq -\frac{1}{2(n-1)} \int_N k_{sc} dV \quad \text{and} \quad H \geq -\frac{1}{n-1} \int_N \bar{k}_R dV$$

(rougher ones but without κ) approach $\frac{3}{4}$ and $\frac{1}{2}$ respectively.

1. The results.

1.1. All manifolds, their boundaries and submanifolds here are supposed to be of class C^∞ unless otherwise stated. Mean, Gauss, Ricci and scalar curvatures are supposed to be defined as in [5, 3.6, 3.7]. Let M be a connected compact n -dimensional, $n \geq 2$, manifold with a nonempty boundary ∂M . Let $\rho(\cdot, \cdot)$ be the distance in M . Put $M(t) = \{p \in M / \rho(p, \partial M) \geq t\}$, $0 \leq t \leq r \stackrel{\text{def}}{=} \max_{q \in M} \rho(q, \partial M)$. Suppose that

- (i) all $M(t)$ are locally convex;
- (ii) normal curvatures of $\partial M(t)$, $0 \leq t < r$, at any of its regular points are not less than $\kappa \geq 0$.

Then M is said to be *convex throughout* or κ -*convex throughout*.

1.2. One could prove that $M(t)$, $t \in [0, r]$, at any of its boundary points, can be "supported by a sphere of curvature κ ". However we will not use this property.

Notice that the conditions (i) and (ii) hold if the normal curvatures of $\partial M(0) = \partial M$ are not less than $\kappa > 0$ and the minimum sectional curvature in M , k_m satisfies

$$(1.2.1) \quad k_m > -\kappa^2.$$

This kind of domain, named normal, was studied in [2, 3 and 4]. The condition (ii) holds for normal domains by Lemma 4 and the subsequent Remark in [2]. The condition (i) can be proved similarly to Lemma 9 in [1]. For that, one should apply Theorem 1(2) from [2] to the geodesic XC constructed in [1].

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At the same time, the domain pasted of a semicircle of unit radius in hyperbolic plane of curvature -1 and a flat figure bounded by a segment of length 2 and the circumferences of radius 2 centered at its ends is $\frac{1}{2}$ -convex throughout but not normal.

1.3. THEOREM. Let m be an n -dimensional, $n \geq 2$, κ -convex throughout domain, V be its volume, $\bar{\kappa}_R(p)$ be the maximum Ricci curvature at $p \in M$, H be the integral mean curvature of ∂M , and G be its integral Gauss curvature. Then

$$(1.3.1) \quad H \geq (n-2)\kappa^2 V - \frac{1}{n-1} \int_M \bar{\kappa}_R dV.$$

Let M_- denote M with nonpositive sectional curvature. Then

$$(1.3.2) \quad G \geq -\kappa^{n-2} \int_{M_-} \bar{\kappa}_R dV.$$

1.4. If the right-hand part of (1.3.1) or (1.3.2) is nonpositive then the estimate is trivial.

For the Euclidean case, (1.3.1) yields

$$(1.4.1) \quad H \geq (n-2)\kappa^2 V$$

which is probably known. Notice however that a combination of the isoperimetric inequality with the fact that the area S of ∂M satisfies $H \geq \kappa S$ yields an estimate rougher than (1.4.1) for large n .

Let M_1 be the metric product of a locally Euclidean 2-dimensional torus and a segment. The equalities in (1.3.1) and (1.3.2) hold when $M = M_- = M_1$. Then all terms in (1.3.1) and (1.3.2) vanish. For domains homeomorphic to a ball, the estimates (1.3.1) and (1.3.2) are also sharp in the following sense. If $M = M_-$ is a ball in 3-dimensional hyperbolic space of curvature -1 then the ratio of the right-hand part of each estimate to its left-hand part (i.e. $(V + \kappa^2 V)/H$ and $2\kappa V/G$ respectively) goes to 1 as the radius of the ball goes to ∞ . The same ratio for the rougher estimate

$$(1.4.2) \quad H \geq -\frac{1}{n-1} \int_M \bar{\kappa}_R dV$$

goes to $\frac{1}{2}$.

1.5. Expressing Ricci curvature in terms of the scalar curvature k_{sc} and sectional curvatures and combining it with (1.3.1) and (1.3.2) one can obtain more estimates. An example follows.

Denote by $v_1, v_2, \dots, v_n$ an orthonormal set of vectors at a point $p \in M$ such that v_n shows in the direction of the maximum Ricci curvature. Let k_{ij} , $1 \leq i < j \leq n$, be the sectional curvature in the plane of the vectors v_i, v_j and let $\underline{k}$ be the minimum sectional curvature at p . Then

$$k_{sc}/2 = \sum_{1 \leq i < j \leq n} k_{ij} = \bar{\kappa}_R + \sum_{1 \leq i < j \leq n-1} k_{ij} \geq \bar{\kappa}_R + (n-1)(n-2)\underline{k}/2.$$

Combining it with (1.3.1) one has

$$H \geq (n-2)\kappa^2 V - \frac{1}{2(n-1)} \int_M k_{sc} dV + \frac{n-2}{2} \int_M \underline{k} dV.$$

If M is a normal domain N then $\underline{k} \geq k_m > -\kappa^2$ (see (1.2.1)) and $\int_N \underline{k} dV \geq -\kappa^2 V$. Then

$$(1.5.1) \quad H \geq \frac{n-2}{2} \kappa^2 V - \frac{1}{2(n-1)} \int_N k_{sc} dV$$

and

$$(1.5.2) \quad H \geq -\frac{1}{2(n-1)} \int_N k_{sc} dV.$$

If N is a ball in 3-dimensional hyperbolic space of curvature -1 then the ratios of the right-hand parts in (1.5.1) and (1.5.2) to their left-hand parts go respectively to 1 and $\frac{3}{4}$ as the radius of the ball goes to ∞ .

2. Curvature of parallel surfaces.

2.1. LEMMA. *Let d be a compact $(n-1)$ -dimensional manifold having a finite number of components. Its boundary ∂d is supposed to be piecewise C^∞ if nonempty. Put $D = d \times [t_1, t_2]$, $0 \leq t_1 < t_2 < r$, and let $f: D \rightarrow M$ be a C^∞ -diffeomorphism such that $S_t \stackrel{\text{def}}{=} f(d \times t) \subset \partial M(t)$ (see 1.1), $t \in [t_1, t_2]$, and $f_*(\partial/\partial t)$ is orthogonal to the hypersurface S_t . Denote by $H(S_t) \geq 0$ and $G(S_t) \geq 0$ the integral mean and the integral Gauss curvature of S_t . (They exist by compactness of S_t .) Then*

$$(2.1.1) \quad \frac{d}{dt} H(S_t) = \frac{1}{n-1} \int_{S_t} \left[\sum_{i=1}^{n-1} k_i - 2 \cdot \sum_{1 \leq i < j \leq n-1} \kappa_i \kappa_j \right] dS,$$

$$(2.1.2) \quad \frac{d}{dt} G(S_t) = \int_{S_t} \sum_{j=1}^{n-1} k_j \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i dS,$$

where $\kappa_i \geq 0$ is the normal curvature of S_t in the i th principal direction and k_i is the sectional curvature in the plane determined by that direction and $f_*(\partial/\partial t)$. (The product $\prod$ in (2.1.2) spreads over $i = 1, 2, \dots, j-1, j+1, \dots, n-1$. For $n = 2$, this product is assumed to be equal to 1 and $\sum \kappa_i \kappa_j$ in (2.1.1) is assumed to be zero.)

2.2. Lemma 2.1 is not new. Variation of the m th curvature of a hypersurface, $1 \leq m \leq n-1$, was investigated in [6] but in a very general form which makes an explicit reference difficult. The following simple calculation seems more appropriate for our case of parallel surfaces.

2.3. PROOF OF LEMMA 2.1. One may assume that $d \subset R^{n-1}$ because in the general case, one can split d into a few pieces each of which is covered by a map of an atlas on d .

Let y^i , $i = 1, 2, \dots, n$, be the coordinates in $f(D)$ borrowed from R^n (with $y^n \in [t_1, t_2]$). Put $Y_1 = f_*(\partial/\partial y^1) \circ f^{-1}$ and $N = Y_n$. Obviously $|N| = 1$, $N \perp Y_i$, $i = 1, 2, \dots, n-1$, and $\nabla_N N = 0$. Put $g_{ij} = \langle Y_i, Y_j \rangle$, $g = \det \|g_{ij}\|$, $c_{ij} = -\langle \nabla_{Y_i} N, Y_j \rangle$, $c = \det \|c_{ij}\|$, $i, j = 1, 2, \dots, n-1$. Then

$$G(S_t) = \int_{S_t} \frac{c}{g} dS = \int_d \frac{c}{\sqrt{g}} dy^1 \dots dy^{n-1},$$

$$H(S_t) = \frac{1}{n-1} \int_{S_t} c_{ij} g^{ij} dS = \frac{1}{n-1} \int_d c_{ij} g^{ij} \sqrt{g} dy^1 \dots dy^{n-1},$$

$$(2.3.1) \quad \frac{d}{dt}G(S_t) = \int_d \left(\frac{c'}{g} - \frac{cg'}{2g^2} \right) \sqrt{g} dy^1 \dots dy^{n-1}$$

$$(2.3.2) \quad \frac{d}{dt}H(S_t) = \frac{1}{n-1} \int_d \left[c'_{ij} g^{ij} + c_{ij} g'^{ij} + c_{ij} g^{ij} \frac{g'}{2g} \right] \sqrt{g} dy^1 \dots dy^{n-1}.$$

Take a point $p \in S_t$. Specialize the coordinates $y^1, \dots, y^{n-1}$ so that $g_{ij} = g^{ij} = \delta_j^i$, $i, j = 1, 2, \dots, n-1$, and the vectors $Y_1, Y_2, \dots, Y_{n-1}$ show in the principal directions of S_t at the point p . Then $c_{ij} = 0$ for $i \neq j$ and $c_{ii} = \kappa_i$ at p .

$$(2.3.3) \quad c = \prod_{i=1}^{n-1} \kappa_i, \quad g = 1, \quad g' = \sum_{i=1}^{n-1} g'_{ii};$$

$$\begin{aligned} c'_{jj} &= -N \langle \nabla_{Y_j} N, Y_j \rangle = -\langle \nabla_N \nabla_{Y_j} N, Y_j \rangle - \langle \nabla_{Y_j} N, \nabla_N Y_j \rangle \\ &= -\langle R(N, Y_j) N, Y_j \rangle + \langle \kappa_j Y_j, \nabla_N Y_j \rangle; \end{aligned}$$

$$(2.3.4) \quad c'_{jj} = k_j + \frac{1}{2} \kappa_j g'_{jj};$$

$$c' = \sum_{j=1}^{n-1} c'_{jj} \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i = \sum_{j=1}^{n-1} k_j \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i + \frac{1}{2} \sum_{j=1}^{n-1} \kappa_j g'_{jj} \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i;$$

$$(2.3.5) \quad c' = \sum_{j=1}^{n-1} k_j \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i + \frac{1}{2} \sum_{j=1}^{n-1} g'_{jj} \cdot \prod_{i=1}^{n-1} \kappa_i.$$

(2.3.3) and (2.3.5) imply

$$\frac{c'}{g} - \frac{cg'}{2g^2} = \sum_{j=1}^{n-1} k_j \cdot \prod_{\substack{1 \leq i \leq n-1 \\ i \neq j}} \kappa_i$$

where the right-hand part does not depend on the choice of the coordinates $y^1, \dots, y^{n-1}$. Therefore, together with (2.3.1), it yields (2.1.2)

$$(2.3.6) \quad g'_{ii} = N \langle Y_i, Y_i \rangle = 2 \langle \nabla_N Y_i, Y_i \rangle = -2c_{ii} = -2\kappa_i \quad \text{at } p.$$

Let G_{ii} be the cofactor of g_{ii} in $\det \|g_{ij}\|$, $i, j = 1, 2, \dots, n-1$. Obviously $G_{ii} = 1$ at p . One can check that $G'_{ii} = g' - g'_{ii}$ at p . Then $g^{ii'} = (G_{ii}/g)' = -g'_{ii} = 2\kappa_i$ at p . Together with (2.3.6), (2.3.3) and (2.3.4), it shows that the square bracket in (2.3.2) is equal to $\sum_{i=1}^{n-1} k_i - 2 \sum_{1 \leq i < j \leq n-1} \kappa_i \kappa_j$ and do not depend on the choice of the coordinates $y^1, y^2, \dots, y^{n-1}$. This implies (2.1.1).

3. Proof of Theorem 1.3.

3.1. Let Ω be the subset of the normal bundle of ∂M such that $v \in \Omega$ if and only if the geodesic $\exp_{\pi(v)} tv$, $0 \leq t \leq 1$, is a shortest path in M between $\exp_{\pi(v)} v$ and ∂M . (π means the natural projection.) The following facts are known, see for example [1, §4].

(i) For any $p \in \partial M$, there exists the maximum number $\phi(p) > 0$ such that $\Omega \supset v$ if $|v| \leq \phi(p)$, $\pi(v) = p$ and v shows inside M .

(ii) The function $\phi(p)$ is continuous in ∂M and

$$0 < \phi(p) \leq r = \max_{q \in M} \rho(q, \partial M).$$

(iii) Ω is compact, homeomorphic to $\partial M \times [0, 1]$ and $\partial\Omega = \{v \in \Omega/v = 0\} \cup \omega$ where $\omega = \{v \in \Omega/|v| = \phi(\pi(v))\}$.

(iv) The mapping $f \stackrel{\text{def}}{=} \exp_{\partial M|_{\Omega \setminus \omega}}$ is a C^∞ -diffeomorphism.

(v) The set $C \stackrel{\text{def}}{=} \exp_{\partial M}(\omega)$ (called cut locus) is compact and has n -dimensional measure zero.

3.2. Fix some $\epsilon \in (0, \min_{p \in \partial M} \phi(p))$. Due to the continuity of ϕ , one can select a C^∞ -function $\phi_\epsilon: \partial M \rightarrow R$ such that $\phi - \epsilon < \phi_\epsilon < \phi$. Put $\Omega_\epsilon = \{v \in \Omega/|v| \leq \phi_\epsilon(\pi(v))\}$. By Sard's Theorem, the set of critical values of ϕ_ϵ has linear measure zero. Therefore for any natural $m > 2$ there exist numbers $0 = t_0 < t_1 < \dots < t_m = \bar{\phi}_\epsilon \stackrel{\text{def}}{=} \max_{p \in \partial M} \phi_\epsilon(p)$ such that $t_i - t_{i-1} < 2\bar{\phi}_\epsilon/m$, $i = 1, 2, \dots, m$, and each t_i , $i \neq m$, is a regular value of ϕ_ϵ . (One can first divide $[0, \bar{\phi}_\epsilon]$ into m equal parts and then "vary a little" the points of division to get rid of the critical values.)

Put $d_i = \{p \in \partial M/\phi_\epsilon(p) \geq t_i\}$, $i = 0, 1, \dots, m-1$. Obviously d_i is nonempty, consists of a finite number of components and is a compact $(n-1)$ -dimensional manifold with a C^∞ -boundary (if any).

3.3. Put $D_i = d_i \times [t_{i-1}, t_i]$ which can be identified with $D_i = \{v \in \Omega_\epsilon/\pi(v) \in d_i, |v| \in [t_{i-1}, t_i]\}$, $1 \leq i \leq m-1$. Put $S_i^i = f(d_i \times t) = \{f(v)/\pi(v) \in d_i, |v| = t\}$ where f is as in 3.1(iv), $1 \leq i \leq m-1$, $t \in [t_{i-1}, t_i]$. Obviously the hypersurface $S_i^i \subset \partial M(t)$ and $f_*(\partial/\partial t)$ is orthogonal to S_i^i . Denote by $H(S_i^i)$ and $G(S_i^i)$ the integral mean and the integral Gauss curvature of S_i^i for the normal $f_*(\partial/\partial t)$ (directed into $M(t)$). They exist by compactness of S_i^i and are nonnegative by local convexity of $\partial M(t)$.

Lemma 2.1 is now applicable. Due to κ -convexity of M , the normal curvatures of S_i^i are $\geq \kappa$. Obviously the sum $\sum_{i=1}^{n-1} k_i$ in (2.1.1) does not exceed $\bar{k}_R$. Then (2.1.1) yields

$$(3.3.1) \quad \frac{d}{dt} H(S_i^i) \leq \frac{1}{n-1} \int_{S_i^i} [\bar{k}_R - (n-2)(n-1)\kappa^2] dS.$$

For $M = M_-$, (2.1.2) yields

$$(3.3.2) \quad \frac{d}{dt} G(S_i^i) \leq \kappa^{n-2} \int_{S_i^i} \bar{k}_R dS.$$

3.4. Integrating (3.3.1) and (3.3.2) over $[t_{i-1}, t_i]$ and denoting volume by $V[\cdot]$, one has

$$(3.4.1) \quad H(S_{t_i}^i) - H(S_{t_{i-1}}^i) \leq \frac{1}{n-1} \int_{f(D_i)} \bar{k}_R dV - (n-2)\kappa^2 V[f(D_i)],$$

$$(3.4.2) \quad G(S_{t_i}^i) - G(S_{t_{i-1}}^i) \leq \kappa^{n-2} \int_{f(D_i)} \bar{k}_R dV \quad (\text{for } M_-).$$

Obviously $S_{t_{i-1}}^i \subset S_{t_i}^{i-1}$ and therefore

$$H(S_{t_{i-1}}^i) \leq H(S_{t_{i-1}}^{i-1}), \quad G(S_{t_{i-1}}^i) \leq G(S_{t_{i-1}}^{i-1}).$$

Neglecting $H(S_{t_{m-1}}^{m-1}) \geq 0$ and $G(S_{t_{m-1}}^{m-1}) \geq 0$, keeping in mind that $S_0^1 = \partial M$ for sufficiently large m and summing up each of (3.4.1) and (3.4.2) for all

$i = 1, 2, \dots, m-1$, one has now

$$(3.4.3) \quad -H \leq \frac{1}{n-1} \int_{P_m} \bar{k}_R dV - (n-2)\kappa^2 V[P_m],$$

$$(3.4.4) \quad -G \leq \kappa^{n-2} \int_{P_m} \bar{k}_R dV \quad (\text{for } M_-)$$

where the "pyramid" $P_m = \bigcup_{i=1}^{m-1} f(D_i)$.

3.5. Let us replace m in turn by $m_q = 2^q m$, $q = 1, 2, \dots$. For each new m_q , the numbers $0 = t_0, t_1, \dots, t_{m_q} = \bar{\phi}_\epsilon$ are supposed to include all such numbers selected earlier. (One can place each new point T of division in the center of each old segment $[t_{i-1}, t_i]$, $i = 1, 2, \dots, 2^{q-1}m$, and then "vary it a little" to avoid critical value but to comply with the condition that each of the segments $[t_{i-1}, T]$, $[T, t_i]$ be shorter than $2\bar{\phi}_\epsilon/2^q m$. This is possible since $t_i - t_{i-1} < 2\bar{\phi}_\epsilon/2^{q-1}m$.) Obviously

$$(3.5.1) \quad P_{m_1} \subset P_{m_2} \subset \dots \subset f(\Omega_\epsilon).$$

3.6. Let $p \in \text{int} f(\Omega_\epsilon)$ and $v = f^{-1}(p)$. Then $|v| < \phi_\epsilon(\pi(v))$. Since $t_i - t_{i-1} < 2\bar{\phi}_\epsilon/m_q$, there exists $i \in \{1, 2, \dots, m_q - 1\}$ such that $t_{i-1} \leq |v| \leq t_i < \phi_\epsilon(\pi(v))$ for sufficiently large m_q . According to 3.2, $\pi(v) \in d_i$ and by 3.3, $v \in D_i$. Thus $p \in f(D_i) \subset P_{m_q}$ and therefore $\bigcup_{q=1}^\infty P_{m_q} \supset \text{int} f(\Omega_\epsilon)$. Together with (3.5.1), it implies that P_m in (3.4.3) and (3.4.4) can be replaced by $f(\Omega_\epsilon)$.

3.7. Denote by C_ϵ the closed ϵ -neighbourhood of the cut locus C . Obviously $f(\Omega_\epsilon) \cup C_\epsilon = M$. Since $|\bar{k}_R| < A = \text{const}$ in M ,

$$\left| \int_M \bar{k}_R dV - \int_{f(\Omega_\epsilon)} \bar{k}_R dV \right| \leq AV[C_\epsilon].$$

By 3.1(v), $V[C_\epsilon] \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore $f(\Omega_\epsilon)$ which appeared in (3.4.3) and (3.4.4) due to 3.6 can be replaced finally by M and M_- respectively. That completes the proof.

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