ORTHOGONAL EXPANSIONS OF VECTORS IN A HILBERT SPACE FOR NON-GAUSSIAN MEASURES

YOSHIAKI OKAZAKI

ABSTRACT. Let $\mathcal{H}$ be a separable Hilbert space and μ a probability Radon measure on $\mathcal{H}$ of second order. Then there exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{H}$ and an O.N.S. $(\xi_n) \subset H$ such that the orthogonal series $\sum_{n=1}^{\infty} a_n \xi_n(x) x_n$ converges in $\mathcal{H}$ μ -almost everywhere and it holds that $x = \sum_{n=1}^{\infty} a_n \xi_n(x) x_n$, μ -almost everywhere, where H is the generating Hilbert space of μ . In the case where μ is a Gaussian measure, a similar result was proved by Kuelbs [2] in general Banach spaces.

1. Let $\mathcal{H}$ be a separable (real or complex) Hilbert space and μ a probability measure on $\mathcal{H}$ of second order, that is, $\int_{\mathcal{H}} \|x\|^2 d\mu(x) < +\infty$. $\mathcal{H}'$ denotes the dual space of $\mathcal{H}$ and is identified with $\mathcal{H}$ by the conjugate linear isometry $\mathcal{H} \ni x \rightarrow (x | \cdot) \in \mathcal{H}'$, where $(\cdot | \cdot)$ is the inner product of $\mathcal{H}$ (Riesz's theorem).

Let $R: \mathcal{H}' \rightarrow L^2(\mu)$ be $R(y) = (y | \cdot)$. We let H be the $L^2(\mu)$ -closure of $R\mathcal{H}'$. Then $R: \mathcal{H}' \rightarrow H$ is a continuous (conjugate) linear mapping since

$$\|R(y)\|^2 = \int |(x | y)|^2 d\mu(x) \leq \left(\int \|x\|^2 d\mu(x) \right) \|y\|^2.$$

THEOREM. *There exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{H}$ and an O.N.S. $(\xi_n) \subset H$ such that*

$$x = \sum_{n=1}^{\infty} a_n \xi_n(x) x_n, \quad \mu\text{-almost everywhere,}$$

where the right-hand side converges in $\mathcal{H}$ for μ -almost every x .

PROOF. First we show that $R: \mathcal{H}' \rightarrow H$ is a Hilbert-Schmidt operator. Let (e_n) be a C.O.N.S. in $\mathcal{H}'$, then we have

$$\sum_{n=1}^{\infty} \|R(e_n)\|^2 = \sum_{n=1}^{\infty} \int |(x | e_n)|^2 d\mu(x) = \int \|x\|^2 d\mu(x) < +\infty,$$

which proves the assertion.

By the well-known representation theorem of a Hilbert-Schmidt operator, there exist $(a_n) \in l^2$, an O.N.S. $(x_n) \subset \mathcal{H}$ and an O.N.S. $(\xi_n) \subset H$ such that

$$R(y)(x) = \sum_{n=1}^{\infty} a_n (x_n | y) \xi_n(x) \quad \text{in } H \subset L^2(\mu).$$

Received by the editors December 1, 1981 and, in revised form, March 5, 1982.

1980 *Mathematics Subject Classification.* Primary 60B11.

Key words and phrases. Orthogonal expansion, generating Hilbert space.

©1982 American Mathematical Society
0002-9939/82/0000-0554/\$01.75

We prove that the orthogonal series $\sum_{n=1}^{\infty} a_n \xi_n(x) x_n$ converges in $\mathcal{H}$ μ -almost everywhere. Since

$$\left\| \sum_{n=k}^m a_n \xi_n(x) x_n \right\|^2 = \sum_{n=k}^m a_n^2 |\xi_n(x)|^2,$$

it is sufficient to show the series $\sum a_n^2 |\xi_n(x)|^2$ converges μ -almost everywhere. By the equality

$$\int \sum_{n=k}^m a_n^2 |\xi_n(x)|^2 d\mu(x) = \sum_{n=k}^m a_n^2$$

and by $(a_n) \in l^2$, it follows that the sequence $\sum_{n=1}^k a_n^2 |\xi_n(x)|^2$, $k = 1, 2, \dots$, converges in $L^1(\mu)$. Hence there is a subsequence which converges μ -almost everywhere. But the sequence is nonnegative and nondecreasing, so the sequence itself converges μ -almost everywhere.

Lastly we show that $x = \sum_{n=1}^{\infty} a_n \xi_n(x) x_n$, μ -almost everywhere. Let (e_k) be a C.O.N.S. in $\mathcal{H}'$. Then we have for μ -almost every x ,

$$\begin{aligned} (x | e_k) &= R(e_k)(x) = \sum_{n=1}^{\infty} a_n (x_n | e_k) \xi_n(x) \\ &= \left(\sum_{n=1}^{\infty} a_n \xi_n(x) x_n | e_k \right), \quad k = 1, 2, \dots \end{aligned}$$

Since (e_k) separates $\mathcal{H}$, we get the assertion.

This completes the proof.

2. The Hilbert space H in the preceding section coincides with the generating Hilbert space of μ constructed by Kuelbs [3, Lemma 2.1]. So by the representation of the mapping R obtained by Kuelbs [3], we can determine $(a_n), (x_n)$ and (ξ_n) explicitly as follows. Let $R^*: H' \rightarrow \mathcal{H}$ be the adjoint linear operator and set $T = R \circ R^*: H' = H \rightarrow \mathcal{H} = \mathcal{H}' \rightarrow H$. Then it holds that

$$R^*(\xi) = \int x \xi(x) d\mu(x)$$

(Bochner integral, Kuelbs [3, (2.1)]),

$$T(\xi)(x) = \int (x | y) \overline{\xi(y)} d\mu(y), \quad \xi \in H$$

(Kuelbs [3, (2.2)]).

Then $T: H \rightarrow H$ (or $L^2(\mu) \rightarrow L^2(\mu)$) is a positive selfadjoint nuclear linear operator (in fact, T is an integral operator with the integrable kernel $(x | y)$, $x, y \in \mathcal{H}$). Let $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < +\infty$ be the eigenvalues and $(\xi_n) \subset H$ the eigenfunctions of T . Put $x_n = c_n^{-1/2} R^*(\xi_n)$. Then we have

$$\begin{aligned} (x_n | x_m) &= c_n^{-1/2} c_m^{-1/2} \left(\int x \xi_n(x) d\mu(x) | \int y \xi_m(y) d\mu(y) \right) \\ &= c_n^{-1/2} c_m^{-1/2} \int \left[\int (x | y) \overline{\xi_m(y)} d\mu(y) \right] \xi_n(x) d\mu(x) \\ &= c_n^{-1/2} c_m^{-1/2} c_m \int \overline{\xi_m(x)} \xi_n(x) d\mu(x) = \delta_{n,m}. \end{aligned}$$

So we get $x = \sum c_n^{-1/2} \xi_n(x) x_n$ μ -almost everywhere. Note that this procedure is very similar to Kuelbs [2, §4].

3. Let ν be the Gaussian Radon measure on $\mathcal{H}$ with the characteristic functional $\exp(-\int |\langle x, x' \rangle|^2 d\mu(x)/2)$, $x' \in \mathcal{H}'$. Then by Kuelbs [2, Theorem 3.1], it follows that $x = \sum a_n \xi_n(x) x_n$ ν -almost everywhere. Kuelbs' arguments are as follows. For the Gaussian measure ν , the orthogonality of (ξ_n) in $L^2(\nu)$ is equivalent to the independence of (ξ_n) as random variables on $(\mathcal{H}, \nu)$. And by the convergence theorem of sums of independent random variables due to Ito and Nisio [1], it follows that $x = \sum a_n \xi_n(x) x_n$ ν -almost everywhere. The purpose of this note is to show $x = \sum a_n \xi_n(x) x_n$ for μ -almost every x . Note that $\nu(A) = 1$ does not imply $\mu(A) = 1$ in general. Note also that the orthogonal random variables (ξ_n) are not independent over $(\mathcal{H}, \mu)$, so we cannot use Ito and Nisio [1]. To prove our theorem the Hilbert-Schmidtness of the operator R is essential. Our result does not hold for general Banach space $\mathcal{H}$.

REFERENCES

1. K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. **5** (1968), 35–48.
2. J. Kuelbs, *Expansions of vectors in a Banach space related to Gaussian measures*, Proc. Amer. Math. Soc. **27** (1971), 364–371.
3. ———, *A strong convergence theorem for Banach space valued random variables*, Ann. Probab. **4** (1976), 744–771.

DEPARTMENT OF MATHEMATICS, KYUSHU UNIVERSITY 33, FUKUOKA 812, JAPAN