TAMING COMPACTA IN E^4

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ABSTRACT. A compactum X in Euclidean 4-space E^4 is shown to be tame if its projection into E^3 is 1-dimensional and if $\dim L \cap X \leq 0$ for each vertical line L in E^4 . An essential ingredient is the result due to J. L. Bryant and D. L. Sumners that a 1-dimensional compactum in a 3-dimensional hyperplane of E^4 is tame in E^4 .

A 0-dimensional or 1-dimensional compact subset X of Euclidean 4-space E^4 is said to be tame in E^4 provided the embedding dimension of X is ≤ 1 ; i.e., for each open cover $\mathcal{U}$ of E^4 and closed 2-dimensional subpolyhedron $K \subset E^4$, there is a self-homeomorphism h of E^4 $\mathcal{U}$ -close to the identity with $h(K) \cap X = \emptyset$. For 0-dimensional compacta and 1-dimensional polyhedra, this notion of tameness is equivalent to the usual one [Br₁, Br₂]. Depending on the notion of tameness to be verified, either the Klee trick [K] or the fact that 1-dimensional subpolyhedra of E^3 can be pushed off 0-dimensional compact subsets of E^3 establishes that a 0-dimensional compact subset of a 3-dimensional hyperplane of E^4 is tame in E^4 . The examples of Bothe [Bo] and McMillan and Row [M-R] of tangled embeddings of 1-dimensional compacta in E^3 necessitated a different argument, supplied by Bryant and Sumners in [Br-S], to establish that a 1-dimensional compact subset of a 3-dimensional hyperplane is tame in E^4 . In this note these results are extended to

THEOREM. *A compact subset $X \subset E^4$ is tame provided, for some projection $p: E^4 \rightarrow E^3$ onto a 3-dimensional hyperplane, $\dim p(X) \leq 1$ and*

$$\dim X \cap p^{-1}(w) \leq 0 \text{ for each } w \in E^3.$$

Appropriate formulations of the result for compact subsets of E^n ($n \neq 4$) that have codimension ≥ 3 are established in [Wr, Theorem 5.3], the principal tool being that codimension ≥ 3 compact subsets of E^n ($n \neq 4$) are tame provided their complements are 1-ULC (see [E] for a thorough discussion). The proof that follows involves applications of the previously mentioned Bryant-Sumners' result interspersed with purely vertical moves to produce homeomorphisms that move 2-complexes off X .

Euclidean n -space is denoted by E^n . The product $E^4 = E^3 \times E^1$ determines a projection map $p: E^4 \rightarrow E^3$. A *partition* of E^1 is a strictly increasing function from the integers to E^1 that is neither bounded above nor below; it is denoted $\{a_i\}$. The partition is said to have mesh δ provided $a_{i+1} - a_i \leq \delta$ for each i . For a subset $W \subset E^3$, we set $W^i = W \times [a_{i-1}, a_i]$. For integers $p < q$, the collection

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$C = \{W^i : p \leq i \leq q\}$ is called a *chain for W with respect to the partition $\{a_i\}$* and the length of C is set equal to $a_q - a_{p-1}$.

LEMMA. If $X \subset E^4$ is a compact subset and $\dim(p^{-1}(s) \cap X) \leq 0$ for some $s \in E^3$, then, for each $\epsilon > 0$, there is a neighborhood $U \subset E^3$ of s and a number $\delta > 0$ such that any chain for U with respect to a partition of mesh δ that has length $\geq \epsilon$ contains an element that does not meet X .

PROOF. If the lemma were false, it would be an easy matter to show that $\dim(p^{-1}(s) \cap X) = 1$. But this would contradict the hypothesis of the lemma, and the lemma is established.

PROOF OF THEOREM. Given $\epsilon > 0$, use the lemma and the compactness of $p(X)$ to produce a finite collection of closed sets $F_1, F_2, \dots, F_m$ that cover E^3 and a $\delta > 0$ such that for each i any chain for F_i with respect to a partition of mesh δ that has length $\geq \epsilon$ contains an element that does not meet X . Assume that $\delta < \epsilon$ and, specifying a partition by setting $a_i = \delta \cdot i$, assume without loss of generality that $X \subseteq E^3 \times [a_0, a_n]$.

Starting with a closed 2-dimensional subpolyhedron $K \subset E^4$, we proceed to produce a self-homeomorphism of E^4 that moves points less than a distance 3ϵ and that moves K off X . The homeomorphism arises as a composition $H_n \circ G_n$ of self-homeomorphisms of E^4 where H_n changes only the E^1 coordinate of points and that by less than 2ϵ and G_n moves points less than distance ϵ . Whenever $(F_j \times [a_i, a_{i+1}]) \cap X = \emptyset$, the center slice $F_j \times \{a_i + \delta/2\}$ is called a *safety zone*. The control on the homeomorphism H_n will be achieved by not permitting H_n to move any point that lies in a safety zone, for the choice of F_j 's establishes a sufficient number of safety zones that H_n cannot move a point a distance $\geq 2\epsilon$.

Set $B_i = p(X \cap (E^3 \times [a_{i-1}, a_i]))$, $A_i = \bigcup \{F_j : F_j \cap B_i = \emptyset\}$, $Y_i = X \cap (E^3 \times [a_0, a_i])$, and $X_i = p(X) \times \{a_i\}$. The homeomorphisms H_n and G_n are the end products of an inductive construction that produces homeomorphisms H_i and G_i of E^4 , $0 \leq i \leq n$, satisfying:

- (1) $H_0 = G_0 = \text{identity homeomorphism}$;
- (2) $H_i \circ G_i(K) \cap Y_i = \emptyset$ for $1 \leq i \leq n$;
- (3) H_i changes only the E^1 coordinate of points and moves no point that lies in a safety zone; and
- (4) G_i moves points less than ϵ .

The inductively constructed homeomorphisms G_i and H_i satisfy

$$G_i^{-1} \circ H_i^{-1}(Y_i) \cap K = \emptyset$$

by (2), where we set $Y_0 = \emptyset$. The sets X_i and, therefore, $G_i^{-1} \circ H_i^{-1}(X_i)$ are tame by [Br-S]. Choose a homeomorphism g of E^4 with $g \circ G_i^{-1} \circ H_i^{-1}(X_i) \cap K = \emptyset$ so close to the identity that $g \circ G_i^{-1} \circ H_i^{-1}(Y_i) \cap K = \emptyset$ and that, setting $G_{i+1} = G_i \circ g^{-1}$, condition (4) is satisfied. The homeomorphism $H_i \circ G_{i+1}$ moves K off Y_i and X_i .

Choose $0 < d < \delta/2$ so that

$$H_i \circ G_{i+1}(K) \cap p(X) \times [a_i - d, a_i] = \emptyset.$$

A homeomorphism h of E^4 equaling the identity outside $E^3 \times [a_i - d, a_{i+1} + \delta/2]$ and on each safety zone and moving only the E^1 coordinate of points is specified as follows. Name a map $\phi: E^3 \rightarrow [a_i, a_{i+1}]$ such that $\phi(A_{i+1}) = a_i$ and $\phi(B_{i+1}) = a_{i+1}$, require that $h(x, a_i - d) = (x, a_i - d)$, $h(x, a_i) = (x, \phi(x))$, and

$h(x, a_{i+1} + \delta/2) = (x, a_{i+1} + \delta/2)$, and extend in the obvious piecewise linear manner. Finally, it is easily checked that G_{i+1} and $H_{i+1} = h \circ H_i$ satisfy the appropriate conditions.

Having established that X has embedding dimension at most one, we conclude that X has dimension at most one; of course, applying [H-W, Theorem V17] to the restriction $p|_X: X \rightarrow p(X)$ immediately reveals the latter.

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