INNER AMENABILITY AND FULLNESS

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ABSTRACT. Let G be a countable group which is not inner amenable. Then the II_1 -factor M is full in the following cases:

- (1) M is given by the group measure space construction from a triple (X, μ, G) with respect to a strongly ergodic measure preserving action of G on a probability space (X, μ) .
- (2) M is the crossed product of a full II_1 -factor by G with respect to an action.

1. Introduction. The set of II_1 -factors is decomposed into two classes. The first is the set of II_1 -factors which have property Γ due to Murray and von Neumann, and which contains the set of hyperfinite II_1 -factors. The second is the set of II_1 -factors which are called full [4]. Most known examples of II_1 -factors are given as one of the following algebras (or composition of those): (1) the group von Neumann algebra associated with an ICC (i.e., infinite conjugacy class) group, (2) the group measure space construction algebra, or more generally, (3) the crossed product of a given von Neumann algebra by an automorphism group.

In [6], Effros introduced the notion "inner amenability" for countable groups. He showed that if a countable ICC group G is not inner amenable then the group factor (associated with G) is full.

In this paper, we shall show that similar results hold for algebras of the above types (2) and (3), so that "noninner amenability" for groups is a desirable property in order to construct a full II_1 -factor.

2. Full II_1 -factor. Let N be a II_1 -factor with the canonical trace τ . Then the following three statements are equivalent [4]: (4) N is full, (5) N has not property Γ , and (6) a (operator norm) bounded sequence (x_n) in N , for which $\|x_n y - y x_n\|_2 \rightarrow 0$ for all $y \in N$, satisfies $\|x_n - \tau(x_n)1\|_2 \rightarrow 0$, where $\|x\|_2 = \tau(x^*x)^{1/2}$ for an $x \in N$.

A countable group G is inner amenable if and only if there is a sequence (ξ_n) in $l^2(G)$ (the Hilbert space of square summable functions on G) such that $\|\xi_n\|_2 = 1$, $\xi_n(1) = 0$ and $\sum_{h \in G} |\xi_n(ghg^{-1}) - \xi_n(h)|^2 \rightarrow 0$ for all $g \in G$, where 1 is the identity of G [6]. The following groups are not inner amenable: (7) the free group with two generators by [6], (8) the free product of two nontrivial groups not both of order 2, and (9) an ICC group with Kazhdan's property T ([8]) by [1].

Received by the editors November 17, 1981 and, in revised form, April 20, 1982.

1980 *Mathematics Subject Classification.* Primary 46L35; Secondary 22D25, 46L10.

Key words and phrases. Factor, group algebra, crossed product, ergodicity.

¹Partially supported by KAKENHI.

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0002-9939/82/0000-0660/\$01.75

Let N be a finite von Neumann algebra acting on a separable Hilbert space H , τ a faithful normal trace on N such that $\tau(1) = 1$, and G a countable group of τ -preserving automorphisms on N . The action of G is said to be *ergodic* on N if $N^G = \{x \in N; g(x) = x \text{ for all } g \in G\} = \mathbb{C}1$, and *strongly ergodic* on N if a bounded sequence (x_n) in N for which $\|g(x_n) - x_n\|_2 \rightarrow 0$ for all $g \in G$, satisfies necessarily $\|x_n - \tau(x_n)1\|_2 \rightarrow 0$ (cf. [5]). A strongly ergodic action is ergodic. If G has property T, then an ergodic action of G is strongly ergodic [2]. Put

$$(\pi(a)\xi)(g) = g^{-1}(a)\xi(g) \quad \text{and} \quad (v(g)\xi)(h) = \xi(g^{-1}h) \\ (a \in N, g, h \in G, \xi \in l^2(G, H)),$$

where $l^2(G, H)$ is the Hilbert space of square summable H -valued functions on G . Then π (resp. v) is a representation of N (resp. G) on $l^2(G, H)$ such that $v(g)\pi(a)v(g)^* = \pi(g(a))$ for all $g \in G$ and $a \in N$. The crossed product M of N by G is the von Neumann algebra generated by $\pi(N)$ and $v(G)$. Let e be the faithful normal expectation of M onto $\pi(N)$ such that $e(v(g)) = 0$ for $g \neq 1$ (see [7], for example). Then M is a finite von Neumann algebra with a faithful normal trace $\tau \cdot e$ and each $x \in M$ has a unique expansion $x = \sum_{g \in G} x(g)v(g)$ ($x(g) \in \pi(N)$ for all $g \in G$) in the sense of $\|\cdot\|_2$ -metric convergence.

It is known that ergodicity is essential to the group measure space construction for a factor. The strong ergodicity is necessary for the group measure space construction of a full II_1 -factor [3].

THEOREM. *Let N be a finite von Neumann algebra with a faithful normal trace τ such that $\tau(1) = 1$, G a countable group of τ -preserving automorphisms of N and M be the crossed product of N by G . Assume that G is not inner amenable. Then*

- (i) *A sequence (x_n) in M , for which $\|x_n\|_2 = 1$ for all n and $\|x_n v(g) - v(g)x_n\|_2 \rightarrow 0$ for all $g \in G$, satisfies $\|x_n - e(x_n)1\|_2 \rightarrow 0$. In particular, $v(G)' \cap M = \pi(N^G)$.*
- (ii) *If the action of G is strongly ergodic on N , then M is a full II_1 -factor.*

PROOF. (i) Let (x_n) be a sequence in M for which $\|x_n\|_2 = 1$ for all n and $\|x_n v(g) - v(g)x_n\|_2 \rightarrow 0$ for all $g \in G$. Let $x_n = \sum_{g \in G} x_n(g)v(g)$ ($x_n(g) \in \pi(N)$) be the Fourier expansion of x_n . For each n , put $\xi_n = \sum_{g \in G} \|x_n(g)\|_2 \delta(g)$, where $\delta(g)$ is the characteristic function of $\{g\}$. Then (ξ_n) is a sequence of unit vectors in $l^2(G)$ and satisfies

$$\begin{aligned} \sum_{h \in G} |\xi_n(ghg^{-1}) - \xi_n(h)|^2 &= \sum_{h \in G} \left| \|x_n(ghg^{-1})\|_2 - \|x_n(h)\|_2 \right|^2 \\ &= \sum_{h \in G} \left| \|x_n(ghg^{-1})\|_2 - \|v(g)x_n(h)v(g)^*\|_2 \right|^2 \\ &\leq \sum_{h \in G} \|x_n(ghg^{-1}) - v(g)x_n(h)v(g)^*\|_2^2 \\ &= \|x_n v(g) - v(g)x_n\|_2^2 \rightarrow 0, \quad \text{for all } g \in G. \end{aligned}$$

If $\|\xi_n - \xi_n(1)\delta(1)\|_2$ does not converge to zero, there are an $\alpha > 0$ and a subsequence $(\xi_{n(k)})$ of (ξ_n) such that $\|\xi_{n(k)} - \xi_{n(k)}(1)\delta(1)\|_2 \geq \alpha$ for all k . For each k , put

$$\zeta_k = (\xi_{n(k)} - \xi_{n(k)}(1)\delta(1)) / \|\xi_{n(k)} - \xi_{n(k)}(1)\delta(1)\|_2.$$

Then the sequence (ζ_k) satisfies that $\|\zeta_k\|_2 = 1$, $\zeta_k(1) = 0$ and

$$\sum_{h \in G} |\zeta_k(ghg^{-1}) - \zeta_k(h)|^2 \rightarrow 0.$$

This contradicts the noninner amenability of G . Hence $\|\xi_n - \xi_n(1)\delta(1)\|_2 \rightarrow 0$. Therefore,

$$\|x_n - e(x_n)\|_2^2 = \sum_{g \neq 1} \|x_n(g)\|_2^2 = \|\xi_n - \xi_n(1)\delta(1)\|_2^2 \rightarrow 0.$$

(ii) If the action of G is ergodic on N , then by (i) $M' \cap M \subset v(G)' \cap M = \pi(N^G) = \mathbf{C}1$. Hence M is a finite factor. If G is not ICC, there is an $h (\neq 1) \in G$ for which $\{ghg^{-1}; g \in G\}$ is a finite set. Put $z = \sum_{g \in G} v(ghg^{-1})$. Then $e(z) = 0$ and $0 \neq z \in v(G)' \cap M = \mathbf{C}1$. This is a contradiction. Hence G is an ICC group.

Therefore M contains a II_1 -factor $v(G)''$, so that M is a II_1 -factor. Assume that M is not full. Since M has property Γ , there is a sequence (x_n) of unitaries in M for which $\|zx_n - x_nz\|_2 \rightarrow 0$ for all $z \in M$ and $\tau(e(x_n)) = 0$ for all n . Since

$$\|x_nv(g) - v(g)x_n\|_2 \rightarrow 0 \quad \text{for all } g \in G,$$

the sequence (x_n) satisfies, by (i), $\|x_n - e(x_n)\|_2 \rightarrow 0$. The expectation e satisfies $e(v(g)yv(g)^*) = v(g)e(y)v(g)^*$ for all $g \in G$ and $y \in M$. Therefore,

$$\|v(g)e(x_n) - e(x_n)v(g)\|_2 \rightarrow 0 \quad \text{for all } g \in G.$$

Since the action of G is strongly ergodic on N , we have that

$$\|e(x_n)\|_2 = \|e(x_n) - \tau(e(x_n))1\|_2 \rightarrow 0.$$

This contradicts the assumption that all x_n are unitaries. Thus M is a full II_1 -factor.

In the case that N is a full II_1 -factor, the assumption of strong ergodicity for the action of G is not necessary.

COROLLARY. *Let N be a full II_1 -factor and G a countable group of automorphisms of N . If G is not inner amenable, then the crossed product M of N by G is a full II_1 -factor.*

PROOF. Let τ be the canonical trace on N . Since N is a factor, by (i) in the Theorem, $M' \cap M \subset \pi(N)' \cap v(G)' \cap M \subset \pi(N)' \cap \pi(N) = \mathbf{C}1$. Hence M is a finite factor. Since N is type II, M is a II_1 -factor. If M is not full, there is a sequence (x_n) of unitaries in M for which $\tau(e(x_n)) = 0$ for all n and $\|x_nz - zx_n\|_2 \rightarrow 0$ for all $z \in M$. By the same proof as (ii) in the Theorem, $\|x_n - e(x_n)\|_2 \rightarrow 0$. On the other hand, the bounded sequence $e(x_n)$ satisfies $\|ye(x_n) - e(x_n)y\|_2 \rightarrow 0$ for all $y \in \pi(N)$, because e is an expectation of M onto $\pi(N)$. Therefore,

$$\|e(x_n)\|_2 = \|e(x_n) - \tau(e(x_n))1\|_2 \rightarrow 0,$$

because N is full. This is a contradiction. Thus M is a full II_1 -factor.

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