DICKSON'S PSEUDODETERMINANT WITHOUT MATRICES

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ABSTRACT. The Dickson pseudodeterminant for orthogonal transformations at characteristic 2 is developed without using its formula in terms of matrix entries.

Suppose F is a field of characteristic 2, V is a finite dimensional vector space over F , and Q is a quadratic form on V , with corresponding bilinear form B . Then B is alternate, so it determines a symplectic geometry on V . We assume that V is nondegenerate relative to B , i.e. that $V^\perp = 0$, so $n = \dim V$ is even, say $n = 2m$. Write $\text{Sp}(V)$ for the symplectic group of B -isometries of V and $O(V)$ for the orthogonal group of Q -isometries. Clearly $O(V) \leq \text{Sp}(V)$.

J. Dieudonné, in [2], used the Clifford algebra $C(V)$ to clarify the relation between two important invariants of V , the Arf invariant (or pseudodiscriminant) and the Dickson invariant (or pseudodeterminant). In the process he shortened considerably the invariance arguments of Arf and Dickson. As part of his argument, however, Dieudonné required the computation of the pseudodeterminant as a function of the entries of a representing matrix for an element of $\text{Sp}(V)$ relative to a symplectic basis. Our purpose is to show that the computation is unnecessary.

Requisite facts about $C(V)$, all of which are elementary, can be found in [3]; also see [1]. We take the point of view that $V \subseteq C(V)$.

Let $\mathfrak{B} = \{u_i, v_i: 1 \leq i \leq m\}$ be a symplectic basis for V . The center of the even subalgebra $C^+(V)$ has as a basis $\{1, z_{\mathfrak{B}}\}$, $z_{\mathfrak{B}} = \sum_i u_i v_i$. An easy calculation yields $z_{\mathfrak{B}}^2 = z_{\mathfrak{B}} + \Delta_{\mathfrak{B}}$, where $\Delta_{\mathfrak{B}} = \sum_i Q(u_i)Q(v_i) \in F$. A change of symplectic basis is effected by choosing $\sigma \in \text{Sp}(V)$ and setting $\sigma\mathfrak{B} = \{\sigma u_i, \sigma v_i: 1 \leq i \leq m\}$.

PROPOSITION 1. *If $\sigma \in \text{Sp}(V)$ then there exists $\delta = \delta_{\mathfrak{B}}(\sigma) \in F$ such that $z_{\sigma\mathfrak{B}} = z_{\mathfrak{B}} + \delta$ and $\Delta_{\sigma\mathfrak{B}} = \Delta_{\mathfrak{B}} + \delta^2 + \delta$. If $\sigma \in O(V)$ then $\delta_{\mathfrak{B}}(\sigma) = 0$ or 1.*

PROOF. Write $z_{\sigma\mathfrak{B}} = \alpha z_{\mathfrak{B}} + \delta$, with $\alpha \in F$ and $\delta = \delta_{\mathfrak{B}}(\sigma) \in F$. Then $\Delta_{\sigma\mathfrak{B}} = z_{\sigma\mathfrak{B}}^2 + z_{\sigma\mathfrak{B}} = \alpha^2 \Delta_{\mathfrak{B}} + \delta^2 + \delta + (\alpha^2 + \alpha)z_{\mathfrak{B}} \in F$, so $\alpha^2 + \alpha = 0$ and $\alpha = 1$, i.e. $z_{\sigma\mathfrak{B}} = z_{\mathfrak{B}} + \delta$. Thus $\Delta_{\sigma\mathfrak{B}} = (z_{\mathfrak{B}} + \delta)^2 + z_{\mathfrak{B}} + \delta = \Delta_{\mathfrak{B}} + \delta^2 + \delta$. If $\sigma \in O(V)$ then $\Delta_{\sigma\mathfrak{B}} = \Delta_{\mathfrak{B}}$, so $\delta^2 + \delta = 0$ and $\delta = 0$ or 1.

Denote by $F^{[2]}$ the additive subgroup $\{\alpha^2 + \alpha: \alpha \in F\}$ of F . By Proposition 1 the coset of $\Delta_{\mathfrak{B}} \bmod F^{[2]}$ is independent of symplectic basis. That coset is called the *Arf invariant* of V .

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PROPOSITION 2. If $\sigma \in O(V)$ and $\tau \in \text{Sp}(V)$ then $\delta_{\mathfrak{B}}(\sigma\tau) = \delta_{\mathfrak{B}}(\sigma) + \delta_{\mathfrak{B}}(\tau)$.

PROOF. There is a unique algebra automorphism $\hat{\sigma}$ of $C(V)$ with $\hat{\sigma}|V = \sigma$. Thus $\hat{\sigma}z_{\mathfrak{B}} = \sum_i (\sigma u_i)(\sigma v_i) = z_{\sigma\mathfrak{B}}$, and so $\hat{\sigma}z_{\tau\mathfrak{B}} = z_{\sigma\tau\mathfrak{B}} = z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\sigma\tau)$, but also $\hat{\sigma}z_{\tau\mathfrak{B}} = \hat{\sigma}(z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\tau)) = z_{\sigma\mathfrak{B}} + \delta_{\mathfrak{B}}(\tau) = z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\sigma) + \delta_{\mathfrak{B}}(\tau)$. Thus $\delta_{\mathfrak{B}}(\sigma\tau) = \delta_{\mathfrak{B}}(\sigma) + \delta_{\mathfrak{B}}(\tau)$.

COROLLARY. Write Z_2 for the prime field $\{0, 1\}$ of F . Then $\delta_{\mathfrak{B}}$ restricts to a homomorphism from $O(V)$ to Z_2 .

We remark that if $\sigma \in O(V)$ is a transvection it is easy to check that $\delta_{\mathfrak{B}}(\sigma) = 1$, using just $z_{\sigma\mathfrak{B}} = z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\sigma)$, so $\delta_{\mathfrak{B}}$ maps $O(V)$ onto Z_2 .

PROPOSITION 3. The restriction of $\delta_{\mathfrak{B}}$ to $O(V)$ is independent of symplectic basis $\mathfrak{B}$.

PROOF. Take $\sigma \in O(V)$ and $\tau \in \text{SP}(V)$. As above $z_{\sigma\tau\mathfrak{B}} = z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\sigma) + \delta_{\mathfrak{B}}(\tau)$, but also $z_{\sigma(\tau\mathfrak{B})} = z_{\tau\mathfrak{B}} + \delta_{\tau\mathfrak{B}}(\sigma) = z_{\mathfrak{B}} + \delta_{\mathfrak{B}}(\tau) + \delta_{\tau\mathfrak{B}}(\sigma)$. Thus $\delta_{\tau\mathfrak{B}}(\sigma) = \delta_{\mathfrak{B}}(\sigma)$.

Write $\delta = \delta_{\mathfrak{B}}|O(V)$. Then δ is Dickson's *pseudodeterminant*; $\ker \delta$ plays the role for $O(V)$ analogous to that of $SO(V)$ when $\text{char } F \neq 2$.

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