THE BAIRE CLASS OF APPROXIMATE SYMMETRIC DERIVATES

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ABSTRACT. It is shown that all approximate symmetric derivatives of measurable functions are in Baire class one. Further, if f is a measurable function which is finite a.e., then its upper and lower approximate symmetric derivatives are in Baire class three.

An extended real-valued function, f , defined on $\mathbf{R}$ is said to be in the first Baire class, abbreviated $\mathfrak{B}_1$, if there is a sequence of continuous functions, $f_1, f_2, \dots$, such that $f_n(x) \rightarrow f(x)$ for all x . In a similar way, f is in the second Baire class, abbreviated $\mathfrak{B}_2$, if there is a sequence of functions, $f_1, f_2, \dots$, each in $\mathfrak{B}_1$, such that $f_n(x) \rightarrow f(x)$ for all x . It is clear that this process can be continued to generate a class of functions, $\mathfrak{B}_\alpha$, corresponding to each ordinal number α (see Goffman [1]). Our purpose here is to show that the upper and lower approximate symmetric derivatives of any measurable function which is finite a.e. are in $\mathfrak{B}_3$ and if it is symmetrically differentiable, then its approximate symmetric derivative is in $\mathfrak{B}_1$.

Let $A \subset \mathbf{R}$. The reflection of A through some $x \in \mathbf{R}$ will be denoted $R_x(A)$. If A is measurable, then its measure is denoted by $|A|$. The upper density of A at $x \in \mathbf{R}$ is written

$$\bar{d}(A, x) = \limsup_{h, k \rightarrow 0+} \frac{|A \cap (x - h, x + k)|}{h + k};$$

the lower density of A at x , $\underline{d}(A, x)$, and the density of A at x , $d(A, x)$, are written similarly.

Let f be a measurable function defined on $\mathbf{R}$. If $x, t \in \mathbf{R}$ such that $t \neq 0$ and $f(x + t) - f(x - t)$ is defined, then we write

$$q(x, t) = \frac{f(x + t) - f(x - t)}{2t}.$$

The upper approximate symmetric derivative of f at x is

$$\bar{f}_{\text{ap}}^s(x) = \inf\{a: \bar{d}(\{t: q(x, t) > a\}, 0) = 0\};$$

the lower approximate symmetric derivative of f at x , $\underline{f}_{\text{ap}}^s(x)$, is defined analogously. If $\bar{f}_{\text{ap}}^s(x) = \underline{f}_{\text{ap}}^s(x)$, whether finite or infinite, then f is approximately symmetrically differentiable at x and the common value of the derivatives is written $f_{\text{ap}}^s(x)$.

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LEMMA. Let f be a finite-valued measurable function defined on $\mathbf{R}$. If $\alpha > 0$, $\beta \in \mathbf{R}$ and $\gamma \in (0, 1)$, then the sets

$$A_f(\alpha, \beta, \gamma) = \{x: |\{t \in (0, \alpha): q(x, t) > \beta\}| > \alpha\gamma\}$$

and

$$B_f(\alpha, \beta, \gamma) = \{x: |\{t \in (0, \alpha): q(x, t) < \beta\}| > \alpha\gamma\}$$

are both open.

PROOF. No generality is lost with the assumption that $\beta = 0$, for if $\beta \neq 0$ we may consider $g(x) = f(x) - \beta x$ in which case

$$A_f(\alpha, \beta, \gamma) = A_g(\alpha, 0, \gamma).$$

Suppose $0 \in A_f(\alpha, 0, \gamma)$ and let C be the set of all $t \in (0, \alpha)$ such that:

(i) f is approximately continuous at t and $-t$ and

(ii) $f(t) > f(-t)$.

Then $|C| > \alpha\gamma$ because of the definition of $A_f(\alpha, 0, \gamma)$ and because the measurability of f implies that it is approximately continuous a. e. (see Goffman [1, p. 190]). For each positive integer, n , define

$$C_n = \{t \in C: f(t) - f(-t) > \frac{1}{n}\}.$$

It is clear that C_n is measurable for each n and that $C = \bigcup_{n=1}^{\infty} C_n$. Therefore there exists an integer, m , large enough so that

$$|C_m| > \alpha\delta > \alpha\gamma$$

for some $\delta > \gamma$. Since f is approximately continuous at t and $-t$ whenever $t \in C_m$, we may choose $r(t) > 0$ such that when $0 < h < r(t)$

$$(1) \quad \left| \left\{ x \in [t-h, t+h]: |f(x) - f(t)| < \frac{1}{2m} \right\} \right| > 2h \left(\frac{3\gamma + 5\delta}{8\delta} \right)$$

and

$$(2) \quad \left| \left\{ x \in [-t-h, -t+h]: |f(x) - f(t)| < \frac{1}{2m} \right\} \right| > 2h \left(\frac{3\gamma + 5\delta}{8\delta} \right).$$

Define

$$\Lambda = \{[t-h, t+h]: t \in C_m, 0 < h < \min\{r(t), t, \alpha-t\}\}.$$

Λ is a Vitali cover for C_m , so there exists a sequence, $\{I_n\} \subset \Lambda$, such that $I_\mu \cap I_\nu = \emptyset$ whenever $\mu \neq \nu$ and

$$\left| C_m - \bigcup_{n=1}^{\infty} I_n \right| = 0.$$

Thus, we may choose an integer, N , large enough so that

$$\left| \bigcup_{n=1}^N I_n \right| > \alpha\delta.$$

Define t_n to be the center of I_n , $J_n = R_0(I_n)$,

$$R_n = \left\{ t \in I_n: |f(t) - f(t_n)| < \frac{1}{2m} \right\}$$

and

$$S_n = \left\{ t \in J_n : |f(t) - f(-t_n)| < \frac{1}{2m} \right\}.$$

According to (1) and (2),

$$(3) \quad |R_n| > |I_n| \frac{3\gamma + 5\delta}{8\delta} \quad \text{and} \quad |S_n| > |J_n| \frac{3\gamma + 5\delta}{8\delta}.$$

Note that if $u \in R_n$ and $v \in S_n$ for some fixed n , then

$$(4) \quad \begin{aligned} f(u) - f(v) &> \left(f(t_n) - \frac{1}{2m} \right) - \left(f(-t_n) + \frac{1}{2m} \right) \\ &= f(t_n) - f(-t_n) - \frac{1}{m} > 0 \end{aligned}$$

because $t_n \in C_m$.

Choose ε such that

$$0 < \varepsilon < \frac{\delta - \gamma}{8\delta} \min_{1 \leq n \leq N} |I_n|$$

and let $|x| < \varepsilon$. Then, using (4) and (3) we see that

$$\begin{aligned} |\{t \in (0, \alpha) : f(x+t) > f(x-t)\}| &\geq \left| \bigcup_{n=1}^N R_x(S_n) \cap R_n \right| \\ &= \sum_{n=1}^N |R_x(S_n) \cap R_n| \\ &\geq \sum_{n=1}^N [|R_x(S_n)| + |R_n| - 2|x| - |I_n|] \\ &> \sum_{n=1}^N \left[|J_n| \left(\frac{3\gamma + 5\delta}{8\delta} \right) + |I_n| \left(\frac{3\gamma + 5\delta}{8\delta} \right) - \left(\frac{\delta - \gamma}{4\delta} \right) |I_n| - |I_n| \right] \\ &= \frac{\gamma}{\delta} \sum_{n=1}^N |I_n| > \alpha\gamma. \end{aligned}$$

Therefore $x \in A_f(\alpha, 0, \gamma)$ and it follows that $(-\varepsilon, \varepsilon) \subset A_f(\alpha, 0, \gamma)$. If $s \in A_f(\alpha, 0, \gamma)$, then by considering $g(x) = f(x + s)$ it can be shown in a manner similar to the above that there is an $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subset A_f(\alpha, 0, \gamma)$. Therefore $A_f(\alpha, 0, \gamma)$ is open.

The proof that $B_f(\alpha, \beta, \gamma)$ is open follows from the observation that $B_f(\alpha, \beta, \gamma) = A_{-f}(\alpha, -\beta, \gamma)$.

THEOREM 1. *If f is a measurable function such that f_{ap}^s exists everywhere, then $f_{\text{ap}}^s \in \mathfrak{B}_1$.*

PROOF. Let $N^+ = \{x : f(x) = \infty\}$ and $N^- = \{x : f(x) = -\infty\}$. If $|N^+| > 0$, then there is an $x \in \mathbf{R}$ such that $d(N^+, x) = 1$. It follows easily that

$$d(\{t : q(x, t) \text{ is defined}\}, 0) = 0$$

which implies that $f_{\text{ap}}^s(x) = -\infty$ and $\bar{f}_{\text{ap}}^s(x) = \infty$. But, $f_{\text{ap}}^s(x)$ exists so this contradiction leads us to conclude that $|N^+| = 0$. It can be shown similarly that $|N^-| = 0$. Therefore f is finite a.e. Since the definition of $f_{\text{ap}}^s(x)$ does not depend upon $f(x)$, no generality is lost with the assumption that f is real-valued everywhere.

Suppose $\beta \in \mathbf{R}$, $\gamma \in (\frac{1}{2}, 1)$, m is a positive integer and

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

Then there is a sequence of integers, $n_1, n_2, \dots$, such that $x \in A_f(\frac{1}{n_i}, \beta - \frac{1}{m}, \gamma)$ for each i . Thus,

$$\left| \left\{ t \in \left(0, \frac{1}{n_i}\right) : q(x, t) > \beta - \frac{1}{m} \right\} \right| > \frac{\gamma}{n_i}$$

and it follows at once that

$$\bar{d}(\{t: q(x, t) > \beta - \frac{1}{m}\}, 0) \geq \gamma > \frac{1}{2}.$$

From the definition of f_{ap}^s we see that $f_{\text{ap}}^s(x) > \beta - \frac{1}{m}$. Therefore

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right) \subset \left\{x: f_{\text{ap}}^s(x) > \beta - \frac{1}{m}\right\},$$

which implies

$$(5) \quad \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right) \subset \{x: f_{\text{ap}}^s(x) \geq \beta\}.$$

Now let $f_{\text{ap}}^s(x) \geq \beta$. Then for each positive integer, m , there is a positive integer, N , such that when $n \geq N$,

$$\left| \left\{ t \in \left(0, \frac{1}{n}\right) : q(x, t) > \beta - \frac{1}{m} \right\} \right| > \frac{\gamma}{n}.$$

Therefore

$$x \in \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right)$$

for all k . This implies that

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

Since $f_{\text{ap}}^s(x) \geq \beta$, this same reasoning holds for any positive integer, m , and so

$$(6) \quad x \in \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

In light of (5) and (6) we conclude that

$$\{x: f_{\text{ap}}^s(x) \geq \beta\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

It follows similarly that

$$\{x: f_{\text{ap}}^s(x) \leq \beta\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_f\left(\frac{1}{n}, \beta + \frac{1}{m}, \gamma\right).$$

Now we apply the lemma to see that both

$$\{x: f_{\text{ap}}^s(x) \geq \beta\} \quad \text{and} \quad \{x: f_{\text{ap}}^s(x) \leq \beta\}$$

are G_δ sets. Using the standard theorem in Goffman [1, p. 141], it follows that $f_{\text{ap}}^s \in \mathfrak{B}_1$.

THEOREM 2. *Let f be a measurable function which is real-valued a. e. and let $\beta \in [-\infty, \infty]$. Then the sets*

$$\{x: \tilde{f}_{\text{ap}}^s(x) > \beta\} \quad \text{and} \quad \{x: \underline{f}_{\text{ap}}^s(x) < \beta\}$$

are both $G_{\delta\sigma}$ sets and the sets

$$\{x: \tilde{f}_{\text{ap}}^s(x) < \beta\} \quad \text{and} \quad \{x: \underline{f}_{\text{ap}}^s(x) > \beta\}$$

are both $F_{\sigma\delta\sigma}$ sets.

PROOF. As before we may assume that f is real-valued everywhere. First, assume $\beta \in \mathbf{R}$. Let $\gamma \in (0, 1)$ and

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f(\alpha, \beta, \gamma).$$

Then for each positive integer, k , there is an $\alpha_k \in (0, \frac{1}{k})$ such that

$$|\{t \in (0, \alpha_k): q(x, t) > \beta\}| > \alpha_k \gamma.$$

Therefore

$$\bar{d}(\{t: q(x, t) > \beta\}, 0) \geq \gamma$$

and we conclude that $\tilde{f}_{\text{ap}}^s(x) > \beta$. From this it follows that if

$$x \in \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right),$$

then $\tilde{f}_{\text{ap}}^s(x) > \beta$. Thus

$$(7) \quad \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right) \subset \{x: \tilde{f}_{\text{ap}}^s(x) > \beta\}.$$

Now, suppose $\tilde{f}_{\text{ap}}^s(x) > \beta$. Then there is a positive integer, n_0 , such that

$$\bar{d}(\{t: q(x, t) > \beta\}, 0) > \frac{1}{n_0}.$$

This implies that for each positive integer, k , there is an $\alpha_k \in (0, \frac{1}{k})$ such that $x \in A_f(\alpha_k, \beta, \frac{1}{n_0})$. Therefore

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n_0}\right) \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right)$$

and so

$$(8) \quad \{x: \tilde{f}_{\text{ap}}^s(x) > \beta\} \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right).$$

From (7) and (8) it follows that

$$\{x: \tilde{f}_{\text{ap}}^s(x) > \beta\} = \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right).$$

It follows from the lemma that $\{x: \tilde{f}_{\text{ap}}^s(x) > \beta\}$ is a $G_{\delta\sigma}$ set.

If $|\beta| = \infty$, then

$$\{x: \tilde{f}_{\text{ap}}^s(x) > \infty\} = \emptyset \quad \text{and} \quad \{x: \tilde{f}_{\text{ap}}^s(x) > -\infty\} = \bigcup_{n=1}^{\infty} \{x: \tilde{f}_{\text{ap}}^s(x) > -n\}$$

are both $G_{\delta\sigma}$ sets. Note that if $\beta \in \mathbf{R}$, then

$$\begin{aligned} \{x: \tilde{f}_{\text{ap}}^s(x) \geq \beta\} &= \bigcap_{n=1}^{\infty} \left\{x: \tilde{f}_{\text{ap}}^s(x) > \beta - \frac{1}{n}\right\}, \\ \{x: \tilde{f}_{\text{ap}}^s(x) \geq \infty\} &= \bigcap_{n=1}^{\infty} \{x: \tilde{f}_{\text{ap}}^s(x) > n\} \end{aligned}$$

and

$$\{x: \tilde{f}_{\text{ap}}^s(x) \geq -\infty\} = \mathbf{R},$$

where all three sets are $G_{\delta\sigma\delta}$ sets. Therefore, for any $\beta \in [-\infty, \infty]$,

$$\{x: \tilde{f}_{\text{ap}}^s(x) < \beta\} = \mathbf{R} - \{x: \tilde{f}_{\text{ap}}^s(x) \geq \beta\}$$

is an $F_{\sigma\delta\sigma}$ set.

To prove the theorem for the lower derivate, it suffices to note that if $g = -f$, then $\underline{f}_{\text{ap}}^s = -(\bar{g}_{\text{ap}}^s)$.

According to Goffman [1, p. 141], Theorem 2 immediately implies the following corollary.

COROLLARY 3. *Let f be a measurable function which is finite a.e. Then $\tilde{f}_{\text{ap}}^s$ and $\underline{f}_{\text{ap}}^s$ are both in $\mathfrak{B}_3$.*

Corollary 3 is an improvement on a theorem of Kundu [2], who showed that if f is continuous, then $\tilde{f}_{\text{ap}}^s$ is measurable. It was shown by Preiss [4] that all ordinary approximate derivatives are in $\mathfrak{B}_1$ and Larson [3] showed that all ordinary symmetric derivatives are in $\mathfrak{B}_1$; Theorem 1 is a partial extension of both of these results.

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