ON MAXIMAL IDEALS DEPENDING ON SOME THIN SETS IN $M(G)$

ENJI SATO

ABSTRACT. Let $M(G)$ be the convolution measure algebra on the LCA group G with dual Γ , and Δ the maximal ideal space of $M(G)$. For $E \subset G$ a compact set, let $Gp(E)$ be the subgroup of G generated algebraically by E , $R(E)$ the measures which are carried by a countable union of translates of $Gp(E)$, and P_E the natural projection from $M(G)$ onto $R(E)$. Also let h_E be the multiplicative linear functional $\mu \mapsto (P_E\mu)(1)$ on $M(G)$. Then we prove that if G is an I -group, and E an H_1 -set, we get $h_E \in \bar{\Gamma}$ (i.e. the closure of Γ in Δ).

1. Introduction. Let G be a nondiscrete LCA group with dual Γ , and $M(G)$ the convolution measure algebra of G (cf. [8, 14]). We denote by $\Delta = \Delta_{M(G)}$ the maximal ideal space of $M(G)$. Given a Borel set E in G , let $I(E)$ be the set of those measures μ in $M(G)$ which satisfy $|\mu|(Gp(E) + x) = 0$ for all $x \in G$, where $Gp(E)$ is the algebraic group generated by E , and $R(E) = I(E)^\perp$ be the set of those measures in $M(G)$ which are singular with respect to all members of $I(E)$. Thus $I(E)$ and $R(E)$ are an L -ideal and an L -subalgebra of $M(G)$, respectively, and $M(G)$ can be decomposed into the direct sum of $I(E)$ and $R(E)$. Moreover, each measure in $R(E)$ is carried by countable union of translates of $Gp(E)$. Let P_E denote the natural projection from $M(G)$ onto $R(E)$. Then P_E is multiplicative and the linear functional $\mu \mapsto (P_E\mu)(1) = (P_E\mu)(G)$ is a complex homomorphism of $M(G)$, which we will denote by h_E .

C. F. Dunkl and D. E. Ramirez [3] proved $h_E \in \bar{\Gamma}$ (i.e. the closure of Γ in Δ) for some Borel set E . In this paper we shall study h_E for some thin sets E .

In order to prove our results, we need some notations. For $\mu \in M(G)$, we denote $\hat{\mu}$ the Gelfand transform of μ which equals to Fourier-Stieltjes transform on Γ , and $\|\hat{\mu}\|_\infty = \sup\{|\hat{\mu}(\gamma)| : \gamma \in \Gamma\}$. Given a set K in G and a natural number n , we define $nK = K + \cdots + K$ (n times). Also we define $nK = \{0\}$ if $n = 0$, and $nK = (-n)(-K)$ if n is a negative integer. Given a subgroup H of G , we shall define that K is H -independent if (a) $K \cap H = \emptyset$, and if (b) whenever $x_1, \dots, x_n$ are finitely many distinct elements of K , $(p_1, \dots, p_n) \in \mathbb{Z}^n$, and $p_1x_1 + \cdots + p_nx_n \in H$, then $p_jx_j \in H$ for all $j = 1, 2, \dots, n$. Notice that when $H = \{0\}$, the above definition agrees with the usual definition (cf. [8]). We shall say that a LCA group G is an I -group if every nonempty open set in G contains an element of infinite order. The author wishes to thank Professor S. Saeki for his many helpful criticisms and suggestions.

Received by the editors August 19, 1981.

1980 *Mathematics Subject Classification*. Primary 43A10; Secondary 43A46.

Key words and phrases. Measure algebra on LCA group, maximal ideal, Helson set.

©1983 American Mathematical Society
 0002-9939/82/0000-0622/\$02.50

2. Maximal ideals and thin sets. Let G be a LCA group and K a compact subset of G . K is called an H_α -set if $\inf\{\|\hat{\mu}\|_\infty \mid \|\mu\| = 1, \mu \in M(K)\} = \alpha$ ($0 < \alpha \leq 1$). We shall simply call K a Helson set, if K is an H_α -set for some α .

THEOREM 1. *Let G be a nondiscrete LCA I-group. Then if K is an H_1 -set, we have $h_K \in \bar{\Gamma}$. Also there exists K and H_α -set ($0 < \alpha < 1$) such that $h_K \notin \bar{\Gamma}$.*

THEOREM 2. *Let G be a nondiscrete LCA group. Then there exists K a non-Helson set in G such that*

$$\lim_{\substack{\gamma \rightarrow \infty \\ \gamma \in \Gamma}} \|\gamma - 1\|_K = 0 \quad \text{and} \quad h_K \notin \bar{\Gamma}.$$

COROLLARY 1 (cf. [1]). *Let G be a nondiscrete LCA I-group. Then there exists $\{h_K\}$ uncountable maximal ideals such that $h_K \in \Delta \setminus \bar{\Gamma}$, and there exist $\{h_K\}$ uncountable maximal ideals such that $h_K \in \bar{\Gamma} \setminus \Gamma$.*

COROLLARY 2. *Under the notation of Theorem 2, h_K is not a strong boundary point for the uniform closure of $M(G)^\wedge$ in $C(\Delta)$.*

The next result is obtained by C. F. Dunkl and D. E. Ramirez [2].

PROPOSITION. *Let f be an element in Δ . Then f is contained in $\bar{\Gamma}$ if and only if $|\hat{\mu}(f)| \leq \|\hat{\mu}\|_\infty$ for all $\mu \in M(G)$.*

In order to prove our results, we need some lemmas.

LEMMA 1. *Let K be an H_1 -set with $K \ni 0$ in G , ϵ a positive number, and E a compact subset in G with $E \cap Gp(K) = \emptyset$. Then there exists a probability measure $\mu_{\epsilon, E} \in M(G)$ such that $\|\hat{\mu}_{\epsilon, E} - 1\|_K < \epsilon$ and $\|\hat{\mu}_{\epsilon, E}\|_E < \epsilon$. Moreover, if we put $A = \{(\epsilon, E) \mid \epsilon > 0, \text{ and } E \text{ is a compact set with } E \cap Gp(K) = \emptyset\}$, and we define $(\epsilon_1, E_1) < (\epsilon_2, E_2)$ for $(\epsilon_i, E_i) \in A$ ($i = 1, 2$), if $\epsilon_2 < \epsilon_1$ and $E_2 \supset E_1$, we obtain that $\lim \mu_{\epsilon, E} = m_\Lambda$ in the w^* -topology of $M(\Gamma_b)$, where Γ_b is the Bohr compactification of Γ , Λ the annihilator of $Gp(K)$ in Γ_b , and m_Λ the normalized Haar measure of Λ .*

PROOF. The first half is obtained by modifying [9, Theorem 5] (cf. [15]). We remark that $\{\mu_{\epsilon, E}\}_A$ is a net in $M(G)$. Then we have the last half by the first half. Q.E.D.

LEMMA 2. *Let K be an H_1 -set containing 0 in G , ϵ a positive number, and F_0 a finite subset of G which is $Gp(K)$ -independent. Also let E_0 be a compact set in G with $E_0 \cap Gp(F_0 \cup K) = \emptyset$. Then there exists ν a probability measure in $M(\Gamma)$ such that $|\hat{\nu} - 1| < \epsilon$ on $F_0 \cup K$ and $|\hat{\nu}| < \epsilon$ on E_0 .*

PROOF. Let m_Λ be as in Lemma 1. By [8], for $\eta > 0$, there exists P_0 a trigonometric polynomial on Λ such that

- (1) $P_0 \geq 0$ on Λ and $\|P_0 m_\Lambda\| = 1$,
- (2) $\text{supp } \hat{P}_0 \subset Gp[F_0 \cup Gp(K)]$,
- (3) $|(P_0 m_\Lambda)^\wedge - 1| < \eta$ on $F_0 + Gp(K)$, and
- (4) $|(P_0 m_\Lambda)^\wedge| < \eta$ on $E_0 + Gp(K)$.

Choose a sufficiently large finite set F with $\{0\} \cup F_0 \subset F \subset Gp(F_0)$ and $(F - F) \cap Gp(K) = \{0\}$, and set $P(\gamma) = \sum_{x \in F} C_x x(\gamma)$ on Γ , where $C_x = \hat{P}_0(x + Gp(K))$ ($x \in F$). For $\mu_{\varepsilon, E}$ as in Lemma 1, we define $\nu_{\varepsilon, E} = |P| \mu_{\varepsilon, E}$. Then by $\Gamma \subset \Gamma_b$ and Lemma 1, we get that

$$(5)' \|\nu_{\varepsilon, E}\| = \int_{\Gamma_b} |P| d\mu_{\varepsilon, E},$$

$$(6)' \left| \int_{\Gamma_b} |P| d\mu_{\varepsilon, E} - \int_{\Gamma_b} |P| dm_{\Lambda} \right| \rightarrow 0 \text{ as } (\varepsilon, E) \mapsto \infty, \text{ and}$$

$$(7)' \|\nu_{\varepsilon, E} - P\mu_{\varepsilon, E}\| = \int_{\Gamma_b} ||P| - P| d\mu_{\varepsilon, E} \mapsto \int_{\Gamma_b} ||P| - P| dm_{\Lambda} \text{ as } (\varepsilon, E) \mapsto \infty.$$

On the other hand, by the definition of P , we have $\int_{\Gamma_b} |P| dm_{\Lambda} = 1$ and $P = |P|$ on Λ . So by (5)', (6)' and (7)', we get that

$$(5) \|\nu_{\varepsilon, E}\| \mapsto 1 \text{ as } (\varepsilon, E) \mapsto \infty, \text{ and}$$

$$(6) \|\nu_{\varepsilon, E} - P\mu_{\varepsilon, E}\| \mapsto 0 \text{ as } (\varepsilon, E) \mapsto \infty.$$

For the above trigonometric polynomial P and for $\eta > 0$, by (5), (6), the assumption of Lemmas 2 and 1, there exists $(\varepsilon, E) \in A$ such that

$$(7) E \supset (\{x_1 - x_2 \mid x_1, x_2 \in F, x_1 \neq x_2\} + K) \cup (E_0 - F),$$

$$(8) E \subset Gp(K) = \emptyset,$$

$$(9) \|\hat{\mu}_{\varepsilon, E}\|_E < \eta / (\sum_{x \in F} |C_x| + 1),$$

$$(10) \|\nu_{\varepsilon, E}\| - 1 < \eta, \text{ and}$$

$$(11) \|\nu_{\varepsilon, E} - P\mu_{\varepsilon, E}\| < \eta.$$

If we define $\nu = \nu_{\varepsilon, E} / \|\nu_{\varepsilon, E}\|$ in $M(\Gamma)$, ν will satisfy the properties of Lemma 2. In fact, for $y \in F_0$ and $k \in K \cup \{0\}$, by the definition of ν , we have that $|\hat{\nu}(y + k) - 1| \leq (|\hat{\nu}_{\varepsilon, E}(y + k) - 1| + |1 - \|\nu_{\varepsilon, E}\||) / \|\nu_{\varepsilon, E}\|$. By (7), (8), (9), (10), and (11), we obtain

$$\begin{aligned} |\hat{\nu}_{\varepsilon, E}(y + k) - 1| &\leq \eta + |(P\mu_{\varepsilon, E})^\wedge(y + k) - 1| \\ &\leq \eta + |C_y \hat{\mu}_{\varepsilon, E}(k) - 1| + \sum_{\substack{x \in F \\ x \neq y}} |C_x| |\hat{\mu}_{\varepsilon, E}(y + k - x)| \\ &\leq \eta + 2\eta + \eta, \end{aligned}$$

and $|\hat{\nu}(y + k) - 1| \leq (5\eta) / (1 - \eta)$. Also for $e \in E_0$, we have that

$$|\hat{\nu}(e)| \leq (\eta + |(P\mu_{\varepsilon, E})^\wedge(e)|) / \|\nu_{\varepsilon, E}\|$$

and $|\hat{\nu}(e)| \leq (2\eta) / (1 - \eta)$ by (7) and (8). Thus ν satisfies what we want for or sufficiently small η . Q.E.D.

PROOF OF THEOREM 1. Let μ be any measure in $M(G)$. By $|\hat{\mu}(h_K)| = |(P_K \mu)^\wedge(1)|$ and the Proposition, it will suffice to show $\|(P_K \mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$ for all $\mu \in M(G)$. For $\varepsilon > 0$, there exists F a finite set such that

$$(1) \|\nu_0 - P_K \mu\| < \varepsilon, \text{ where } \nu_0 = \mu|_{Gp(K) + F}.$$

Let $k \in K$ be fixed. Since K is an H_1 -set, $K - \{k\}$ is an H_1 -set with $K - \{k\} \ni 0$ in G . Then there exists F_0 a finite set in G such that F_0 is $Gp(K - \{k\})$ -independent and $Gp(F \cup K) = Gp(F_0 \cup (K - \{k\}))$. So there exists N a natural number such that

$$(2) \|\tau - \nu_0\| < \varepsilon, \text{ where } \tau = \mu|_{K_N}, \text{ and}$$

$$K_N = N(F_0 \cup (K - \{k\})) \cup (-F_0) \cup (-K + \{k\}).$$

Now let E_0 be a compact subset of G such that

(3) $\|(\mu - \tau)|_{G \setminus E_0}\| < \varepsilon$ and $E_0 \cap Gp(F_0 \cup (K - \{k\})) = \emptyset$.

By Lemma 2, there exists ν a probability measure in $M(\Gamma)$ such that

(4) $\|\hat{\nu} - 1\|_{F_0 \cup (K - \{k\})} < \varepsilon/N$, and

(5) $\|\hat{\nu}\|_{E_0} < \varepsilon$.

By (1), (2), and (3), we get that

$$\left| \int_G \hat{\nu}(x) \gamma(x) d\mu(x) \right| \geq \left| \int_{K_N} \hat{\nu}(x) \gamma(x) d\mu(x) \right| - \left| \int_{E_0} \hat{\nu}(x) \gamma(x) d\mu(x) \right| - 3\varepsilon.$$

Also by (4) and (5), we have that $|\hat{\nu}(x) - 1| < \varepsilon$ for $x \in K_N$, and

$$\left| \int_G \hat{\nu}(x) \gamma(x) d\mu(x) \right| < |\hat{\tau}(\gamma)| - \varepsilon - \varepsilon - 3\varepsilon.$$

On the other hand, we have that $\left| \int_G \hat{\nu}(x) \gamma(x) d\mu(x) \right| \leq \|\hat{\mu}\|_\infty \|\nu\| = \|\hat{\mu}\|_\infty$. Then we get that $|\hat{\tau}(\gamma)| \leq \|\hat{\mu}\|_\infty + 5\varepsilon$. Thus by (1) and (2), we have that $|(P_K \mu)(\gamma)| \leq \|\hat{\mu}\|_\infty + 7\varepsilon$ and $\|(P_K \mu)\|_\infty \leq \|\hat{\mu}\|_\infty$.

The last half is as follows. By Körner [6], there exist K a perfect independent H_α -set ($0 < \alpha < 1$) and a probability measure $\sigma \in M(K)$ such that $\lim_{\gamma \rightarrow \infty} |\hat{\sigma}(\gamma)| < 1$. If we assume $h_K \in \bar{\Gamma}$, there exist $\{\gamma_n\}$ such that $\lim_{n \rightarrow \infty} \int \gamma_n d\sigma = \int h_K d\sigma$. Since we have that

$$\lim_{n \rightarrow \infty} \left| \int \gamma_n d\sigma \right| = \overline{\lim}_{\gamma \rightarrow \infty} |\sigma(\gamma)| < 1$$

and $\int h_K d\sigma = 1$ by the property of σ , we get the contradiction. Q.E.D.

LEMMA 3 [7, p. 114]. *Let G be a nondiscrete LCA group, and L a totally disconnected perfect independent set and non-Helson set in G . Then there exists $K \subset L$ a clopen non-Helson set in the relative topology of L such that $\|\hat{\mu}|_{Gp(K)}\|_\infty \geq 3\|\hat{\mu}\|_\infty$ for some nonzero measure μ in $M(L)$.*

LEMMA 4 [9]. *Let G be a nondiscrete LCA group, $F \subset G$ a finite set, and $E \subset G$ a closed subset with $E \cap Gp(F) = \emptyset$. Then to each $\varepsilon > 0$, there exists $f \in A(G) = L^1(\Gamma)$ such that $0 \leq f \leq 1$ on G , $f = 1$ on F , $f = 0$ on some neighborhood of E , and $\|f\|_{A(G)} < 1 + \varepsilon$.*

PROOF OF THEOREM 2. Since G is a nondiscrete LCA group, there exists L a totally disconnected perfect independent set and non-Helson set such that

$$\overline{\lim}_{\substack{\gamma \rightarrow \infty \\ \gamma \in \Gamma}} \|\gamma - 1\|_L = 0$$

(cf. [5, 8]). Then by Lemma 3, there exists $K \subset L$ a perfect subset with the property of Lemma 3. Now we shall show $h_K \notin \bar{\Gamma}$. By the Proposition, it is sufficient to show that we get a contradiction if we assume $\|(P_K \nu)\|_\infty \|\hat{\nu}\|_\infty$ for all $\nu \in M(G)$. We define $K' = L \setminus K$, and for μ as in Lemma 3, $P_K \mu = \mu_1 + \mu_2$ where $\mu_1 = \mu|_K$ and $\mu_2 = \mu|_{K'}$. Since L is an independent set, we have that (a) $\mu_1 = \mu|_{Gp(K)}$, and (b) $\mu_2 \in M_d(K')$. By (a) and the choice of μ , we get $\|\hat{\mu}_1\|_\infty = \|\hat{\mu}|_{Gp(K)}\|_\infty \geq 3\|\hat{\mu}\|_\infty$.

Also by (b), we assume that the support of μ_2 is a finite set. By Lemma 4, to each $\varepsilon > 0$, there exists $f \in A(G)$ ($0 \leq f \leq 1$) such that $f = 1$ on $\text{supp } \mu_2$, $f = 0$ on K , and $\|f\|_{A(G)} < 1 + \varepsilon$. Then for $\gamma \in \Gamma$, we have that

$$\int_G \gamma f d(P_K \mu) = \int_K \gamma f d(P_K \mu) + \int_{K^c} \gamma f d(P_K \mu) = \int_{K^c} \gamma f d(P_K \mu).$$

Thus we get that

$$\begin{aligned} \left| \int_G \gamma d\mu_2 \right| &= \left| \int_{K^c} \gamma f d(P_K \mu) \right| = \left| \int_G \gamma f d(P_K \mu) \right| \\ &\leq \|f\|_{A(G)} \|(P_K \mu)^\wedge\|_\infty < (1 + \varepsilon) \|(P_K \mu)^\wedge\|_\infty, \end{aligned}$$

and $\|\hat{\mu}_2\|_\infty \leq \|(P_K \mu)^\wedge\|_\infty (\leq \|\hat{\mu}\|_\infty)$. On the other hand, by the assumption of the contradiction and the property of Lemma 3, we have that

$$\begin{aligned} \|\hat{\mu}\|_\infty &\geq \|\hat{\mu}_1\|_\infty - \|\hat{\mu}_2\|_\infty = \|\hat{\mu}_{|Gp(K)}\|_\infty - \|\hat{\mu}_2\|_\infty \\ &\geq 3\|\hat{\mu}\|_\infty - \|\hat{\mu}\|_\infty = 2\|\hat{\mu}\|_\infty. \end{aligned}$$

Therefore we get a required contradiction. Q.E.D.

PROOF OF COROLLARY 1. By the well-known fact [7], there exists K a totally disconnected perfect independent set with $\sigma(\neq 0) \in M(K)$ and $\hat{\sigma}(\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$. Then K is divided with $K = K_1 \cup K_2$ such that $K_1 \cap K_2 \neq \emptyset$ and K_1, K_2 are nonempty perfect compact subsets with $\tau(\neq 0) \in M(K_1)$ and $\hat{\tau}(\gamma) \mapsto 0$ as $\gamma \mapsto \infty$. Since K is an independent set, $\{K_1 + \{x\}\}_{x \in K_2}$ are pairwise disjoint sets. By [11], there exists $K(x)$ a totally disconnected perfect independent and non-Helson set with $K(x) \subset K_1 + \{x\}$. Then $\{K(x)\}_{x \in K_2}$ are an uncountable set, and $\{h_{K(x)}\}_{x \in K_2}$ are what we require. In fact, we assume $x \neq y$ and $K(x) \cap (Gp(K(y)) + z) \neq \emptyset$. Hence, if there exist $w, w' \in K(x)$ such that $w = z + \sum_i n_i k(y, i)$ and $w' = z + \sum_i n'_i k'(y, i)$ where n_i, n'_i integers and $k(y, i), k'(y, i) \in K(y)$, we have $w - w' = -\sum_i n'_i k_i(y, i) + \sum_i n_i k(y, i)$. Since we have $Gp(K(x)) \cap Gp(K(y)) = \{0\}$ by $x \neq y$, we get $w = w'$.

Thus there exists μ a continuous probability measure on $K(x)$ such that $\hat{\mu}(h_{K(y)}) = 0$. Therefore we proved the first half. Also it is well known that there exists K a totally disconnected perfect independent H_1 -set in G . So the last half can be proved in the same way as the above method by Theorem 2. Q.E.D.

The proof of Corollary 2 is clear by Theorem 2 and [13].

3. H -Kronecker set. In this section, we show a generalization of [12] in some sense. First we define H -Kronecker set.

DEFINITION. Let G be a nondiscrete LCA group, K a compact set, and H a closed subgroup of G . K is called an H -Kronecker set if, to each $\varepsilon > 0$ and $f \in C(K)$ with $|f| = 1$, there exists $\gamma \in H^\perp$ such that $\|f - \gamma\|_K < \varepsilon$. We simply call K a Kronecker set, if $H = \{0\}$.

Next we shall show the existence of a nontrivial H -Kronecker set.

THEOREM 3. *Let G be a metric I -group, and H a σ -compact closed subgroup with Haar measure zero such that $\{px: x \in U\} \not\subset H$ for any natural number p and for any neighborhood U of $0 \in G$. Then there exists K an H -Kronecker set which is a totally disconnected perfect set.*

The proof is obtained by modifying [8, 5.2].

THEOREM 4. *Under the above assumption, let ρ be a nonzero continuous measure on K . Then we have that for any $\gamma \in \Gamma$, $\mu_i \in M(G)$ ($1 \leq i \leq n$), and $\varepsilon > 0$, there exists $\gamma_0 \in \Gamma$ such that $|(h_H\mu_i)^\wedge(\gamma) - (h_H\mu_i)^\wedge(\gamma_0)| < \varepsilon$ ($1 \leq i \leq n$) and $|\hat{\rho}(\gamma_0)| > \|\rho\|/2$.*

PROOF. Let π be the natural mapping from G to G/H , and $\tilde{\pi}: M(G) \mapsto M(G/H)$ the mapping induced by π . Then we remark $\tilde{\pi}(\mu)^\wedge = \hat{\mu}|_{H^\perp}$ for all $\mu \in M(G)$. Now by the definition of K , $\pi(K)$ is a Kronecker set on G/H . Hence, by an application of [12] in G/H , there exists $\gamma_0 \in H^\perp$ such that

$$|\tilde{\pi}(\bar{\gamma}h_H\mu_i)^\wedge(1) - \tilde{\pi}(\bar{\gamma}h_H\mu_i)^\wedge(\gamma_0)| < \varepsilon \quad (1 \leq i \leq n),$$

and $|\tilde{\pi}(\bar{\gamma}\rho)^\wedge(\gamma_0)| > \|\rho\|/2$ (cf. [4]).

That is, we have

$$|(h_H\mu_i)^\wedge(\gamma) - (h_H\mu_i)^\wedge(\gamma_0 + \gamma)| < \varepsilon \quad (1 \leq i \leq n),$$

and $|\hat{\rho}(\gamma_0 + \gamma)| > \|\rho\|/2$. This establishes the result. Q.E.D.

REFERENCES

1. G. Brown, *Idempotents in the closure of the characters*, Bull. London Math. Soc. **4** (1972), 43–46.
2. C. F. Dunkl and D. E. Ramirez, *Locally compact subgroup of the spectrum of the measure algebra*, Semigroup Forum **3** (1971), 95–107.
3. ———, *Bounded projections on Fourier-Stieltjes transform*, Proc. Amer. Math. Soc. **31** (1972), 122–126.
4. I. Glicksberg and I. Wik, *The range of Fourier-Stieltjes transforms of parts of measures*, Conference on Harmonic Analysis, Maryland (1971), Lecture Notes in Math., vol. 266, Springer-Verlag, Berlin and New York, 1972.
5. C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Springer-Verlag, Berlin and New York, 1979.
6. T. W. Körner, *Some results on Kronecker, Dirichlet and Helson sets. II*, J. Analyse Math. **27** (1974), 260–388.
7. L. A. Lindahl and F. Poulsen (editors), *Thin sets in harmonic analysis*, Marcel Dekker, New York, 1972.
8. W. Rudin, *Fourier analysis on groups*, Interscience Tracts No. 12, Interscience, New York, 1962.
9. S. Saeki, *On the union of two Helson sets*, J. Math. Soc. Japan **23** (1971), 636–648.
10. ———, *On strong Ditkin sets*, Ark. Mat. **10** (1972), 1–7.
11. ———, *Symmetric maximal ideals in $M(G)$* , Pacific J. Math. **54** (1974), 229–243.
12. ———, *Bohr compactification and continuous measure*, Proc. Amer. Math. Soc. **80** (1980), 244–246.
13. S. Saeki and E. Sato, *Critical points and point derivations on $M(G)$* , Michigan Math. J. **25** (1978), 147–161.
14. J. L. Taylor, *Measure algebras*, CBMS Regional Conf. Ser. in Math., no. 16, Amer. Math. Soc., Providence, R.I., 1973.
15. N. Th. Varopoulos, *Groups of continuous functions in harmonic analysis*, Acta Math. **125** (1970), 109–152.