SPACES WHOSE CLOSED IMAGES ARE M_1

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ABSTRACT. Almost local finiteness is introduced. The class of all spaces with a σ -almost locally finite base is an intermediate class between that of free L -spaces and that of M_1 -spaces. The class is countably productive, hereditary and the closed image of a space in the class is M_1 .

1. Introduction. The M_i -spaces, $i = 1, 2, 3$, were defined in 1961 by J. Ceder [3] as natural generalizations of metrizable spaces. C. R. Borges [1] later showed that the class of M_3 -spaces, which he renamed stratifiable spaces, was a particular well-behaved class of spaces. The equivalence of M_3 and M_2 was proved independently by G. Gruenhage [4] and H. Junnila [8].

The following is one of the most outstanding problems in general topology: "Does M_3 imply M_1 ?"

The following are equivalent to this problem by the recent result of R. Heath and H. Junnila [6]:

"Is the closed (perfect) image of an M_1 -space M_1 ?"

"Is every (closed) subspace of an M_1 -space M_1 ?"

It is well known that the closed image of an M_3 -space is M_3 . The perfect irreducible image of an M_1 -space was shown to be M_1 by C. R. Borges and D. J. Lutzer [2].

On the other hand, free L -spaces were introduced by K. Nagami [10] for discussing the dimension theory. A space is a free L -space if and only if it is embedded in the countable product of L -spaces. Every Lašnev space is an L -space and every free L -space is an M_1 -space.

In this paper we first introduce the notion of almost local finiteness as a generalization of local finiteness. Then the class of all spaces with a σ -almost locally finite base is investigated. This new class is an intermediate class between that of free L -spaces and that of M_1 -spaces. This class contains every space with a σ -closure preserving base of clopen sets which R. Heath and H. Junnila call " M_0 -spaces". The class is countably productive, hereditary and, in particular, the closed image of a space in the class is M_1 .

All spaces are assumed to be regular T_1 . Neighborhoods need not be open. The letter N denotes the set of all natural numbers.

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2. Almost local finiteness. In this section we introduce the notion of almost local finiteness as a generalization of local finiteness. The basic properties are studied.

DEFINITION 2.1. Let X be a space, x a point of X , and $\mathcal{A}$ a family of subsets of X . $\mathcal{A}$ is said to be *almost locally finite at x* if there exist a neighborhood U of x and a finite family $\mathfrak{B}$ of subsets of X such that $\mathcal{A}|U = \{A \cap U: A \in \mathcal{A}\} \subset \{B \cap V: B \in \mathfrak{B} \text{ and } V \text{ is a neighborhood of } x\}$. $\mathcal{A}$ is said to be *almost locally finite in X* if $\mathcal{A}$ is almost locally finite at every point of X .

DEFINITION 2.2. Let X be a space, x a point of X , and $\mathcal{A}$ a family of subsets of X . $\mathcal{A}$ is said to be *closure preserving at x* if for any $\mathcal{A}' \subset \mathcal{A}$, $x \in \text{Cl } \bigcup \{A: A \in \mathcal{A}'\}$ implies $x \in \bigcup \{\text{Cl } A: A \in \mathcal{A}'\}$. $\mathcal{A}$ is said to be *closure preserving in X* if $\mathcal{A}$ is closure preserving at every point of X .

EXAMPLE 2.3. Let X be a space and x a point of X . The following three are typical examples of an almost locally finite family at x :

- (1) a family which is locally finite at x ;
- (2) a family of neighborhoods of x ;
- (3) a closure preserving open family $\mathcal{A}$ such that for each $A \in \mathcal{A}$, either $x \in A$ or $x \notin \text{Cl } A$. (Let $U = X \setminus \bigcup \{\text{Cl } A: x \notin \text{Cl } A, A \in \mathcal{A}\}$ and $\mathfrak{B} = \{X\}$.) In particular, a family of clopen sets which is closure preserving at x .

PROPOSITION 2.4. Let X be a space, x a point of X and $\mathcal{A}$ an almost locally finite family at x . Then $\mathcal{A}$ is closure preserving at x .

PROOF. Let $\mathcal{A}' \subset \mathcal{A}$ and $x \in \text{Cl } \bigcup \{A: A \in \mathcal{A}'\}$. By Definition 2.1, there exist a neighborhood U of x and a finite family $\mathfrak{B}$ such that $\mathcal{A}'|U \subset \{B \cap V: B \in \mathfrak{B} \text{ and } V \text{ is a neighborhood of } x\}$. Put $\mathfrak{B}' = \{B \in \mathfrak{B}: B \cap V \in \mathcal{A}' \text{ for some neighborhood } V \text{ of } x\}$. Then $x \in \text{Cl } \bigcup \{B: B \in \mathfrak{B}'\}$. Since $\mathfrak{B}'$ is finite, $x \in \text{Cl } B_0$ for some $B_0 \in \mathfrak{B}'$. By the definition of $\mathfrak{B}'$, there exist a member A_0 of $\mathcal{A}'$ and a neighborhood V_0 of x such that $B_0 \cap V_0 = A_0 \cap U$. Then $x \in \text{Cl}(B_0 \cap V_0) = \text{Cl}(A_0 \cap U) \subset \text{Cl } A_0$, which completes the proof.

EXAMPLE 2.5. There exists a closure preserving family which is not almost locally finite. Let $X = D \cup \{\infty\}$ be a one point compactification of an infinite discrete space D . Then the family $\mathcal{A} = \{\{d, \infty\}: d \in D\}$ is closure preserving in X but not almost locally finite at the point ∞ .

PROPOSITION 2.6. Let X be a space, $x \in X$ and $\mathcal{A}_\lambda$ an almost locally finite family at x for each $\lambda \in \Lambda$. Let $A_\lambda \supset \bigcup \{A: A \in \mathcal{A}_\lambda\}$. If $\{A_\lambda: \lambda \in \Lambda\}$ is locally finite at x , then the family $\{A: A \in \mathcal{A}_\lambda \text{ for some } \lambda \in \Lambda\}$ is almost locally finite at x .

REMARK. The local finiteness of $\{A_\lambda: \lambda \in \Lambda\}$ in this proposition cannot be weakened to almost local finiteness. For example let $X = D \cup \{\infty\}$ be a one point compactification of an infinite discrete space D . For each $d \in D$, let $\mathcal{A}_d = \{\{d\}, X \setminus \{d\}\}$ and $A_d = X$. Then $A_d \supset \bigcup \{A: A \in \mathcal{A}_d\}$ and $\{A_d: d \in D\}$ is almost locally finite in X . But the family $\{A: A \in \mathcal{A}_d \text{ for some } d \in D\}$ is not closure preserving, hence not almost locally finite at ∞ .

The proofs of the following two results are straightforward and are thus omitted.

PROPOSITION 2.7. *Let X and Y be spaces, $x \in X$, $y \in Y$ and let $\mathcal{A}$ be an almost locally finite family at x and $\mathcal{B}$ an almost locally finite family at y . Then the product $\mathcal{A} \times \mathcal{B} = \{A \times B: A \in \mathcal{A}, B \in \mathcal{B}\}$ is almost locally finite at (x, y) .*

PROPOSITION 2.8. *Let X be a space, $x \in X$ and $\mathcal{A}$ and $\mathcal{B}$ almost locally finite families at x . Then:*

- (1) *for any subset C of X , $\mathcal{A}|C$ is almost locally finite at x ;*
- (2) *$\mathcal{A} \wedge \mathcal{B} = \{A \cap B: A \in \mathcal{A}, B \in \mathcal{B}\}$ is almost locally finite at x ;*
- (3) *let $\mathcal{A}_\lambda \subset \mathcal{A}$ for each $\lambda \in \Lambda$ and put $A_\lambda = \bigcup \mathcal{A}_\lambda$. Then the family $\{A_\lambda: \lambda \in \Lambda\}$ is almost locally finite at x .*

3. Spaces with a σ -almost locally finite base. Now we are going to study the class of all spaces with a σ -almost locally finite base. Lemma 2.2 of K. Nagami [11] suggests this notion to us. We start with theorems asserting that the class is countably productive and hereditary.

THEOREM 3.1. *Let X_n be a space with a σ -almost locally finite base for each $n \in N$. Then the product $X = \prod_{n=1}^{\infty} X_n$ has a σ -almost locally finite base.*

PROOF. Let $\bigcup \{\mathcal{B}_{n,m}: m \in N\}$ be a σ -almost locally finite base of X_n . For each $k \in N$ and $(n_1, n_2, \dots, n_k) \in N^k$, put

$$\mathcal{W}(n_1, \dots, n_k) = \left\{ B_1 \times \cdots \times B_k \times \prod_{j=k+1}^{\infty} X_j: B_i \in \mathcal{B}_{i,n_i}, 1 \leq i \leq k \right\}.$$

Then by Proposition 2.7, each $\mathcal{W}(n_1, \dots, n_k)$ is almost locally finite in X . Hence $\bigcup \{\mathcal{W}(n_1, \dots, n_k): (n_1, \dots, n_k) \in N^k, k \in N\}$ is a σ -almost locally finite base of X .

THEOREM 3.2. *Let X be a space with a σ -almost locally finite base and $Y \subset X$. Then the subspace Y has a σ -almost locally finite base.*

PROOF. This follows immediately from Proposition 2.8(1).

THEOREM 3.3. *The following implications hold: Free L -space $\xrightarrow{(1)}$ Space with a σ -almost locally finite base $\xrightarrow{(2)}$ M_1 -space.*

PROOF. (2) is obvious from Proposition 2.4.

Let us prove (1). Since every free L -space can be embedded into a countable product of L -spaces, it is enough to show that every L -space has a σ -almost locally finite base. (See [9] for the definition of L -spaces.)

Let X be an L -space and $\bigcup \{\mathcal{F}_n: n \in N\}$ a σ -discrete closed network of X . For each $n \in N$, let $\{D(F): F \in \mathcal{F}_n\}$ be a discrete family of open sets such that $F \subset D(F)$. Let $\mathcal{U}_F$ be an approaching anticover of F which is locally finite in $X \setminus F$. Put

$$\mathcal{B}(F) = \{B(\mathcal{U}) = F \cup (\bigcup \{U: U \in \mathcal{U}\}): \mathcal{U} \subset \mathcal{U}_F, B(\mathcal{U}) \subset D(F) \text{ and } B(\mathcal{U}) \text{ is open}\}.$$

Then, by Proposition 2.8(3), $\mathfrak{B}(F)$ is almost locally finite at every point of $X \setminus F$. Since every $B \in \mathfrak{B}(F)$ is a neighborhood of F , $\mathfrak{B}(F)$ is almost locally finite at every point of F . Therefore $\mathfrak{B}(F)$ is almost locally finite in X . By Proposition 2.6, $\mathfrak{B}_n = \bigcup \{\mathfrak{B}(F) : F \in \mathfrak{F}_n\}$ is almost locally finite in X . Hence $\bigcup \{\mathfrak{B}_n : n \in N\}$ is σ -almost locally finite and is obviously a base of X .

THEOREM 3.4. *Let X be a space with a σ -almost locally finite base and F a closed subset of X . Then F has an almost locally finite neighborhood base of open sets.*

PROOF. Let $\bigcup \{\mathfrak{B}_n : n \in N\}$ be a σ -almost locally finite base of X . According to Theorem 3.3, X is M_1 , which implies that X is perfectly normal. So we can take a family $\{G_n : n \in N\}$ of open sets such that $G_1 \supset \text{Cl } G_2 \supset G_2 \supset \text{Cl } G_3 \supset \cdots$, $\bigcap \{G_n : n \in N\} = F$. Put $\mathfrak{V}_n = \mathfrak{B}_n|G_n$. By Proposition 2.8(1), $\mathfrak{V}_n$ is almost locally finite. Observe that $\{G_n : n \in N\}$ is locally finite at every point of $X \setminus F$. By Proposition 2.6, $\mathfrak{V} = \bigcup \{\mathfrak{V}_n : n \in N\}$ is almost locally finite at every point of $X \setminus F$. Put

$$\mathfrak{W} = \{W(\mathfrak{V}) = \bigcup \{V : V \in \mathfrak{V}\} : \mathfrak{V}' \subset \mathfrak{V}, F \subset W(\mathfrak{V}')\}.$$

Then by Proposition 2.8(3), $\mathfrak{W}$ is almost locally finite at every point of $X \setminus F$. Since every $W \in \mathfrak{W}$ is a neighborhood of F , $\mathfrak{W}$ is almost locally finite at every point of F . Hence $\mathfrak{W}$ is almost locally finite in X . Obviously $\mathfrak{W}$ constitutes a neighborhood base of open sets of F and the proof is completed.

Now we are in a position to show that the closed image of a space with a σ -almost locally finite base is M_1 . We need the following theorem of G. Gruenhage.

THEOREM 3.5 [5]. *Suppose X is stratifiable and has the following property:*

(*) *Whenever H and K are closed subsets of X with $H \subset K$, then H has a σ -closure preserving outer base in K .*

Then every closed image of X has the same property and is therefore M_1 .

THEOREM 3.6. *Let $f: X \rightarrow Y$ be a closed onto map and let X have a σ -almost locally finite base. Then Y is an M_1 -space.*

PROOF. In the light of Theorem 3.5, it is enough to show that every space X with a σ -almost locally finite base has property (*). Apply Theorems 3.2, 3.4 and Proposition 2.4.

We do not know the answer to the following question:

Question 3.7. Does every perfect (closed) image of a space with a σ -almost locally finite base have a σ -almost locally finite base?

However, by Theorem 3.4, we have

COROLLARY 3.8. *Let X be a space with a σ -almost locally finite base and F a closed subset. Then X/F has a σ -almost locally finite base.*

Now let us ask whether any of the implications of Theorem 3.3 reverse.

EXAMPLE 3.9. There exists a space with a σ -almost locally finite base which is not free L . In [7], M. Itô constructed a free L -space Y containing a closed set E such that

Y/E is not free L . By Corollary 3.8, Y/E has a σ -almost locally finite base. Thus Y/E is the desired example.

The following question remains unanswered:

Question 3.10. Is there an M_1 -space which does not have a σ -almost locally finite base?

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