A CHARACTERIZATION OF UNCONDITIONED WEAK SEQUENCES

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ABSTRACT. Unconditioned weak sequences are characterized in terms of the Koszulhomology and applications to the theory of Buchsbaum rings and in a special case to the resolution of the ideal generated by the weak sequence are considered.

Introduction. In the theory of local rings there has been a rapid development of the theory of Buchsbaum rings, for basic facts see [8, 4, 5, 6]. Buchsbaum rings may be defined by the property that for every system of parameters $\mathfrak{x}$ of the local ring $(R, \mathfrak{m})$ the difference of $\text{length}(R/(\mathfrak{x}))$ and multiplicity of the ideal $(\mathfrak{x})$ is an invariant, i.e. does not depend on the choice of the system of parameters $\mathfrak{x}$, or by the property: every system of parameters $x_1, \dots, x_n$ is a weak sequence, i.e.

$$(*) \quad \mathfrak{m}((x_1, \dots, x_{i-1}): x_i) \subseteq (x_1, \dots, x_{i-1}) \quad \text{for all } 1 \leq i \leq n.$$

In the following we will say that a sequence of elements $x_1, \dots, x_n$ is an unconditioned weak sequence if it satisfies $(*)$ for any renumbering. Hence every system of parameters in a Buchsbaum ring is certainly an unconditioned weak sequence.

Another characterization of Buchsbaum rings follows from the result in [7], saying: for every system of parameters $\mathfrak{x}$ the Koszulhomology $H_*(\mathfrak{x})$ is a vector-space, equivalently $H_1(\mathfrak{x})$ is a vectorspace. Hence it is equivalent demanding the weak sequence property for all systems of parameters $\mathfrak{x}$ of a local ring and demanding $H_*(\mathfrak{x})$ ($H_1(\mathfrak{x})$) is a vectorspace for all such $\mathfrak{x}$. So one may ask: given any sequence $x_1, \dots, x_n$ in a local ring, what does it mean for the Koszulhomology if we demand $\mathfrak{x}$ to be an unconditioned weak sequence?

In this paper we want to answer this question. It turns out that the assumption for $\mathfrak{x}$ to be an unconditioned weak sequence does not imply in general that the Koszulhomology modules $H_i(\mathfrak{x})$ for $i \geq 1$ are vectorspaces, but that $H_i(\mathfrak{x})$ modulo a certain submodule is a vectorspace. These submodules vanish e.g. if $\mathfrak{x}$ has the additional property to be unconditioned relatively regular in the sense of Fiorenzini [1]. For a system of parameters $\mathfrak{x}$ of a Buchsbaum ring this property of course holds, since $\mathfrak{x}$ is even a (unconditioned) d -sequence (see [2]).

Further we show that for an unconditioned weak sequence $\mathfrak{x}$ the Koszulcomplex is always a (module-)direct summand of the minimal R -free resolution of $R/(\mathfrak{x})$ for $n \leq 2$. We then show that if for every multi-index $(\nu_1, \dots, \nu_n)$ the syzygetic part of

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the Koszulhomology $\tilde{H}_i(x_1^{\nu_1}, \dots, x_n^{\nu_n}; M)$ is a vectorspace for a finitely generated R -module M , then $x_1^{\nu_1}, \dots, x_n^{\nu_n}$ is a weak sequence on M for all $(\nu_1, \dots, \nu_n)$. Hence from this fact and our characterization of weak sequences the characterization of Buchsbaum rings derived from Suzuki [7] follows and, more generally, together with the result in [7], this gives another proof of Shimoda's characterization of Buchsbaum modules (cf. [3]).

Finally we consider, in the case that the $H_i(\mathfrak{x})$ are vectorspaces, the resolution of $R/(x_1, x_2)$ as an R -module for an unconditioned weak sequence x_1, x_2 and give a condition when this resolution is obtained by the Koszulcomplex and copies of the resolutions of the residue class field suitably lifted.

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Notations. All rings $(R, \mathfrak{m})$ will be commutative with 1, noetherian and local with maximal ideal $\mathfrak{m}$. Let $x_1, \dots, x_n$ be elements of R . We will write $\mathfrak{x}$ for the whole sequence $x_1, \dots, x_n$; $(\mathfrak{x})$ denotes the R -ideal generated by $\mathfrak{x}$. For any subset $I \subseteq \{1, \dots, n\}$ and any $i \in I$ we write I_i for the subset of all elements of I excluding i . Also $x_1, \dots, \hat{x}_i, \dots, x_n$ stands for the sequence with x_i omitted.

Let $I \subseteq \{1, \dots, n\}$, then the Koszulcomplex $K := K((x_i)_{i \in I}; M)$ for a R -module M is the R -complex whose underlying graded module is $\bigwedge(\bigoplus_{i \in I} RT_i) \otimes_R M$ and the basiselements T_i map to x_i for $i \in I$. If $J = \{j_1, \dots, j_s\} \subseteq I$ with $j_1 < \dots < j_s$ the element T_J denotes the product $T_{j_1} \cdots T_{j_s} \in K((x_i)_{i \in I}; R)$. For our convenience we will also use the index J for the coefficient of the basiselement T_J . $Z_k(K)$, $B_k(K)$ denote the submodule of cycles and boundaries of K_k respectively. For the homology modules $H_i(K(\mathfrak{x}; M))$ we will also write $H_i(\mathfrak{x}; M)$ for short.

Main section.

LEMMA 1. *Let $(R, \mathfrak{m})$ be a local ring and $x_1, \dots, x_n$ a sequence of elements of R such that $(x_1, \dots, \hat{x}_i, \dots, x_n) \subsetneq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$; then $Z_k(K(\mathfrak{x}, R)) \subseteq \mathfrak{m}K_k(\mathfrak{x}, R)$ for all $k \geq 1$.*

PROOF. Assume the assertion does not hold, so there is an $I \subseteq \{1, \dots, n\}$ such that

$$T_I - \sum_{J \neq I} \lambda_J T_J \in Z(K) \quad (\lambda_J \in R),$$

i.e.

$$\delta(T_I) - \sum_{J \neq I} \lambda_J \delta(T_J) = 0,$$

hence

$$\sum_{i \in I} (-1) \cdots x_i T_{I_i} - \sum_{J \neq I} \lambda_J \sum_{j \in J} (-1) \cdots x_j T_{J_j} = 0.$$

So comparing coefficients of the same basiselements we obtain the inclusion of ideals $(x_i)_{i \in I} \subseteq (x_j)_{j \notin I}$, a contradiction to our assumption.

LEMMA 2. *Let $(R, \mathfrak{m})$ be a local ring, $x_1, x_2 \in R$ be an unconditioned weak sequence with $(x_i) \subsetneq (x_1, x_2)$ for $i = 1, 2$, then the Koszulcomplex $K(x_1, x_2; R)$ is a (module-) direct summand of the minimal R -free resolution of $R/(x_1, x_2)$.*

PROOF. As $K_0(x_1, x_2) = R$, K_1 is the free module $K_1 = RT_1 \oplus RT_2$ with basis-elements T_1, T_2 which map to x_1, x_2 respectively, we only have to show that $x_2T_1 - x_1T_2 \notin mZ_1(K)$.

So assume $z = x_2T_1 - x_1T_2 \in mZ_1(K)$, then we have a representation $z = \sum \alpha_i z_i$, $\alpha_i \in m$, $z_i \in Z_1(K)$. We write $z_i = a_iT_1 - b_iT_2$, $a_i, b_i \in R$, then $\sum \alpha_i a_i = x_2$, $\sum \alpha_i b_i = x_1$. Since $z_i \in Z_1(K)$, we have $b_i \in (x_1 : x_2)$ for all i and therefore $\alpha_i b_i = \delta_i x_1$ for some $\delta_i \in R$, as x_1, x_2 form a weak sequence. Using $\sum \alpha_i b_i = x_1$, we obtain that there is an i_0 such that $\delta_{i_0} \notin m$, hence a unit in R . Consider $w := \delta_{i_0}^{-1} z_{i_0} = aT_1 - bT_2 \in Z_1(K)$, put $\alpha := \alpha_{i_0}$, then

$$\alpha w - (x_2T_1 - x_1T_2) = (\alpha a - x_2)T_1 \in mZ_1(K).$$

Now $w = aT_1 - bT_2$ is a cycle, so $a \in (x_2 : x_1)$, hence $\alpha a \in (x_2)$. So we may write $\alpha w - (x_2T_1 - x_1T_2) = \mu x_2T_1$ with some $\mu \in R$. Therefore $\mu x_1 x_2 = 0$, and by the weak sequence property we derive $\mu m^2 = 0$.

If $x_1 \notin m^2$ or $x_2 \notin m^2$, then certainly $z = x_2T_1 - x_1T_2 \notin m^2 K \supseteq mZ_1(K)$ a contradiction to our assumption. Hence we may assume that $x_1, x_2 \in m^2$, so $x_2\mu \in m^2\mu = 0$ and therefore $\alpha w - (x_2T_1 - x_1T_2) = 0$. So $\alpha a = x_2$ and $\alpha b = x_1$.

As $b \in (x_1 : x_2)$, we have $mb \subseteq (x_1)$. Now consider the multiplication map $b \cdot : m \rightarrow (x_1)$ given by the multiplication with b . We have the exact sequence

$$0 \rightarrow \text{Ker}(b \cdot) \rightarrow m \xrightarrow{b \cdot} (x_1) \rightarrow 0.$$

Tensoring with R/m , we have a surjective map $m/m^2 \rightarrow (x_1)/m(x_1)$. Since $\alpha b = x_1$, $\alpha \in m \setminus m^2$. Let $m_1, \dots, m_n$ denote a minimal system of generators of m with $m_1 = \alpha$ and $m_i b \in m(x_1)$ for $i = 2, \dots, n$. Now write $m_i b = \gamma_i x_1$ with $\gamma_i \in m$ for $i = 2, \dots, n$; then $\alpha \gamma_i - m_i \in \text{Ker}(b \cdot)$, hence $\alpha \gamma_i - m_i \in (0 : x_1) = (0 : m)$ for $i = 2, \dots, n$.

So for all $i \geq 2, j \geq 1$ we have $m_j(\alpha \gamma_i - m_i) = 0$, hence $m_i m_j \in m^3 \cap (\alpha)$, therefore m^2/m^3 is generated by the class of $m_1^2 = \alpha^2$, so m^2 is generated by α^2 by Nakayama's Lemma, hence m^k is generated by α^k for all $k \geq 2$. So we may write $x_1 = u_1 \alpha^{n_1}$, $x_2 = u_2 \alpha^{n_2}$ with u_1, u_2 units in R and n_1, n_2 natural numbers. Since $n_1 \leq n_2$ or $n_2 \leq n_1$ we have $(x_1, x_2) = (x_1)$ or $(x_1, x_2) = (x_2)$ the desired contradiction.

PROPOSITION 1. Let (R, m) be a local ring, $x_1, \dots, x_n$ be an unconditioned weak sequence such that $(x_1, \dots, \hat{x}_i, \dots, x_n) \subseteq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$, then the elements $d(T_I)$ for $I \subseteq \{1, \dots, n\}$, $\#I \geq 2$ form part of a basis of $Z(K)/mZ(K)$, where $K = K(\mathfrak{x}, R)$.

PROOF. Let $I \subseteq \{1, \dots, n\}$ and denote

$$z = \sum_{i \in I} (-1)^{\sigma(i, I)} x_i T_i^{\wedge},$$

where $\sigma(i, I) = \#\{j \in I : j < i\}$, then we have to show $z \notin mZ(K)$.

Let $R \rightarrow \bar{R} := R/(x_j)_{j \notin I}$ be the canonical surjection, which induces a homomorphism $K(\mathfrak{x}, R) \rightarrow K(\mathfrak{x}, \bar{R})$. We reduce $K(\mathfrak{x}, \bar{R})$ modulo the ideal $\langle T_j \rangle_{j \notin I}$ and obtain a homomorphism of differential graded algebras

$$K(x_1, \dots, x_n; R) \rightarrow K((x_i)_{i \in I}; R/(x_j)_{j \notin I}).$$

Now it is enough to show that the image of z under this homomorphism is not contained in $\bar{m}K((x_i)_{i \in I}; R/(x_j)_{j \notin I})$, where $\bar{m}$ denotes the maximal ideal of $R/(x_j)_{j \notin I}$.

So without loss of generality by renumbering we only have to show

$$z = \sum_i (-1)^{i+1} x_i T_1 \cdot \dots \cdot \hat{T}_i \cdot \dots \cdot T_n \notin \bar{m} Z_{n-1}(K).$$

Let us assume this is not true and reduce again modulo $x_3, \dots, x_n$, then

$$\bar{z} = \bar{x}_1 T_2 \cdot \dots \cdot T_n - \bar{x}_2 T_1 T_3 \cdot \dots \cdot T_n \in \bar{m} Z_{n-1}(K(\bar{x}, R/(x_3, \dots, x_n))),$$

hence there exist $\alpha_j \in \bar{m}$, $z_j \in Z_{n-1}(K(\bar{x}, R/(x_3, \dots, x_n)))$, $z_j = \sum_i (-1)^{i+1} a_{ji} T_1 \cdot \dots \cdot \hat{T}_i \cdot \dots \cdot T_n$, such that $z = \sum_j \alpha_j z_j$. Since $\delta z_j = 0$, we have $a_{j1} x_2 T_3 \cdot \dots \cdot T_n - a_{j2} x_1 T_3 \cdot \dots \cdot T_n = 0$, hence $a_{j1} x_2 - a_{j2} x_1 = 0$ for all j and $\sum_j \alpha_j a_{j1} = \bar{x}_1$, $\sum_j \alpha_j a_{j2} = \bar{x}_2$. So consider $w_j := a_{j1} T_1 - a_{j2} T_2 \in Z_1(K(\bar{x}_1, \bar{x}_2; R/(x_3, \dots, x_n)))$, then

$$\sum_j \alpha_j w_j = \bar{x}_2 T_1 - \bar{x}_1 T_2 \in \bar{m} Z(K(\bar{x}_1, \bar{x}_2; R/(x_3, \dots, x_n))),$$

which contradicts Lemma 2. This proves the assertion of the Proposition.

REMARK. There exist unconditioned weak sequences $\bar{x}$, for which the assumption $(x_1, \dots, \hat{x}_i, \dots, x_n) \subsetneq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$ does not hold and hence the assertion of the proposition cannot be true in these cases.

EXAMPLE. Let $(R, \bar{m})$ be a regular local ring of dimension 1 and let x be a generator of $\bar{m}$. Consider $x_1 := x$, $x_2 := x^r$ ($r \geq 1$). Then $(0 : x_1) = (0 : x_2) = 0$, $(x_1 : x_2) = R$, hence $\bar{m}(x_1 : x_2) = (x_1)$, $(x_2 : x_1) = (x^{r-1})$, hence $\bar{m}(x_2 : x_1) = (x_2)$.

LEMMA 3. Let $(R, \bar{m})$ be a local ring, $x_1, \dots, x_n$ an unconditioned weak sequence such that $(x_1, \dots, \hat{x}_i, \dots, x_n) \subsetneq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$. Let $I \subseteq \{1, \dots, n\}$ and denote $z = \sum_{i \in I} \alpha_i T_i \in K_1(\bar{x}, \bar{R})$ a cycle, then for every $i \in I$ and for every $\beta \in \bar{m}$ we have $\beta \alpha_i \in \bar{m}(x_j)_{j \in I^c}$.

PROOF. By a suitable renumbering we may assume $I = \{1, \dots, s\}$. Modulo the ideal $(x_3, \dots, x_n)$ we consider $\bar{z} = \bar{\alpha}_1 T_1 + \bar{\alpha}_2 T_2$. $\bar{z}$ is a cycle in $K_1(\bar{x}, R/(x_3, \dots, x_n))$. Now it is enough to show that $\beta \alpha_1 \in \bar{m}(x_2)$ and $\beta \alpha_2 \in \bar{m}(x_1)$ modulo $(x_3, \dots, x_n)$ for every $\beta \in \bar{m}$. So without loss of generality $n = 2$, $z = aT_1 - bT_2$ and we show $\beta a \in \bar{m}(x_2)$, $\beta b \in \bar{m}(x_1)$ for $\beta \in \bar{m}$.

So take any $\beta \in \bar{m}$, then there exist $u_1, u_2 \in R$ such that $\beta a = u_2 x_2$, $\beta b = u_1 x_1$ by the weak sequence property.

Now u_1, u_2 cannot simultaneously be units in R . If so, we have $u_1 x_1 x_2 = u_2 x_1 x_2$, since $z = aT_1 - bT_2$ is a cycle, and hence by the weak sequence property $(u_1 - u_2)\bar{m}^2 = 0$. So $\alpha = u_2^{-1} u_1 \alpha$ for every $\alpha \in \bar{m}^2$. Now since x_1, x_2 is an unconditioned weak sequence, so is x'_1, x'_2 , where we denote $x'_1 := u_1 x_1$, $x'_2 := u_2 x_2$. Consider the cycle $z' := u_2 a T'_1 - u_1 b T'_2 \in K(x'_1, x'_2; R)$. Then

$$\begin{aligned} u_2^{-1} \beta z' &= \beta a T'_1 - u_2^{-1} u_1 \beta b T'_2 = \beta a T'_1 - \beta b T'_2 \\ &= x'_2 T'_1 - x'_1 T'_2 \in \bar{m} Z(K), \end{aligned}$$

which contradicts the assertion of Lemma 2.

So at least one of the coefficients u_1, u_2 is a nonunit. Let $u_1 \in \bar{m}$. We have to show that also $u_2 \in \bar{m}$. So assume u_2 is a unit, without loss of generality $u_2 = 1$. Then

$x_1x_2 = \beta ax_1 = \beta bx_2 = u_1x_1x_2$, hence $x_1x_2 = 0$. We obtain $x_2 \in (0 : x_1) = (0 : m)$, therefore $x_2m = 0$ hence $(0 : x_2) = m$. But then we have $x_2 = \beta a \in m^2 = m(0 : x_2) = m(0 : m) = 0$ a contradiction.

COROLLARY. *Let (R, m) be a local ring, $x_1, \dots, x_n$ an unconditioned weak sequence such that $(x_1, \dots, \hat{x}_i, \dots, x_n) \subsetneq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$, then $m((x_j)_{j \in I_i^*} : x_i) \subseteq m(x_j)_{j \in I_i^*}$.*

PROOF. Let $\alpha \in ((x_j)_{j \in I_i^*} : x_i)$, $\beta \in m$, then there exist $\gamma_j \in R$ for $j \in I_i^*$ such that $\alpha x_i = \sum_{j \in I_i^*} \gamma_j x_j$. Then $\alpha T_i - \sum_{j \in I_i^*} \gamma_j T_j$ is a cycle in $K(\mathfrak{x}, R)$, hence by Lemma 3 we have $\beta \alpha \in m(x_j)_{j \in I_i^*}$.

We are now able to prove the following

THEOREM. *Let (R, m) be a local ring, $x_1, \dots, x_n$ a sequence of elements such that $(x_1, \dots, \hat{x}_i, \dots, x_n) \subsetneq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$, then the following are equivalent:*

- (1) $x_1, \dots, x_n$ is an unconditioned weak sequence,
- (2) for all $I \subseteq \{1, \dots, n\}$, $1 \leq k \leq \#I$, $J_0 \subseteq I$ with $\#J_0 = \#I - k$:

$$H_k(K(x_i)_{i \in I}) / \iota \left(\left(m(\mathfrak{x}) \sum_{\substack{\#J=k \\ J \cap J_0 \neq \emptyset}} K_k(x_j)_{j \in J} \right) \cap Z_k(K(x_i)_{i \in I}) \right)$$

is a vectorspace,

- (3) for all $I \subseteq \{1, \dots, n\}$, $i_0 \in I$:

$$H_1(K(x_i)_{i \in I}) / \iota \left((m(\mathfrak{x}) K_1(x_i)_{i \in I_{i_0}}) \cap Z(K(x_i)_{i \in I}) \right)$$

is a vectorspace,

- (4) for all $I \subseteq \{1, \dots, n\}$, $i_0 \in I$:

$$H_1(K(x_i)_{i \in I}) / H_1(K(x_i)_{i \in I_{i_0}})$$

is a vectorspace.

(ι denotes the canonical map $Z_*(K) \rightarrow H_*(K)$.)

PROOF. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1). Consider the long exact homology sequence associated to

$$0 \rightarrow K(x_i)_{i \in I_{i_0}} \rightarrow K(x_i)_{i \in I} \rightarrow K(x_i)_{i \in I_{i_0}}[-1] \rightarrow 0,$$

then we obtain that the vectorspace $H_1(K(x_i)_{i \in I}) / H_1(K(x_i)_{i \in I_{i_0}})$ is just the kernel of the multiplication map

$$R / (x_i)_{i \in I_{i_0}} \xrightarrow{x_{i_0}} R / (x_i)_{i \in I_{i_0}},$$

hence $m((x_i)_{i \in I_{i_0}} : x_i) \subseteq (x_i)_{i \in I_{i_0}}^*$.

(1) $\Rightarrow$ (2). Let $\#I = r$, $1 \leq k \leq r$ and $J_0 \subseteq I$ with $\#J_0 = r - k$. Consider

$$z = \sum_{\substack{\#R=k \\ R \subseteq I}} \alpha_R T_R \in Z_k(K(x_i)_{i \in I}).$$

Denote $\mathfrak{N}_0 := I \setminus J_0$, then $\alpha_{\mathfrak{N}_0} \in ((x_j)_{j \notin \mathfrak{N}_0} : (x_{j'})_{j' \in \mathfrak{N}_0})$, so for any $\beta \in \mathfrak{m}$: $\beta \alpha_{\mathfrak{N}_0} \in \mathfrak{m}(x_j)_{j \notin \mathfrak{N}_0}$ by the corollary of Lemma 3. So write

$$\beta \alpha_{\mathfrak{N}_0} = \sum_{j \notin \mathfrak{N}_0} \gamma_j x_j \quad \text{with } \gamma_j \in \mathfrak{m},$$

hence

$$\beta z - \delta \left(\sum_{j \notin \mathfrak{N}_0} \gamma_j T_j T_{\mathfrak{N}_0} \right) \in \left(\mathfrak{m}(\mathfrak{x}) \sum_{\substack{\#J=k \\ J \cap J_0 \neq \emptyset}} K_k(x_j)_{j \in J} \right) \cap Z_k(K(x_i)_{i \in I}),$$

since for each $\mathfrak{N}$ we also have $\beta \alpha_{\mathfrak{N}} \in \mathfrak{m}(\mathfrak{x})$ again by the corollary of Lemma 3.

REMARK 1. If $x_1, \dots, x_n$ is an unconditioned relatively $\mathfrak{m}$ regular sequence with respect to $\mathfrak{m}(\mathfrak{x})$ in the sense of Fiorenini [1], then $\mathfrak{m}(\mathfrak{x})K_k(x_i)_{i \in I} \cap Z_k(K(x_i)_{i \in I}) \subseteq \mathfrak{m}B_k(K(x_i)_{i \in I})$ for all $I \subseteq \{1, \dots, n\}$ by Fiorenini's result, hence the submodules by which we have to divide $H_k(K(x_i)_{i \in I})$ in (2) and (3) in the theorem vanish.

For a system of parameters $\mathfrak{x}$ of a Buchsbaum ring this assumption certainly holds, since in this case $\mathfrak{x}$ is even a d -sequence (cf. [2]).

REMARK 2. In general for unconditioned weak sequences the homology modules are not vectorspaces.

EXAMPLE. Let $A = k[[X_1, X_2, Y_1, Y_2, Z]]$ be the powerseries ring in five variables over a field k and denote $\mathfrak{m}$ the maximal ideal of A .

Let

$$R := A / (Y_1^2, Y_1 Y_2, Y_2^2, Y_1 X_2 - Y_2 X_1, Y_1 m^2, Y_2 m^2, Z^2, Z m^2, Z Y_1, Z Y_2 - Y_1 X_2),$$

denote $\mathfrak{m}$ the maximal ideal of R and x_1, x_2, y_1, y_2, z the residues of the corresponding elements of A . Then

(a) x_1, x_2 is an unconditioned weak sequence.

(b) $H_1(x_1, x_2; R)$ is not a vector space.

PROOF. (a) $(0 : x_1) = (0 : x_2) = (y_1, y_2, z)\mathfrak{m}$,

$$(x_1 : x_2) = (y_1, y_2, z)\mathfrak{m} + (y_1, x_1),$$

$$(x_2 : x_1) = (y_1, y_2, z)\mathfrak{m} + (y_2, x_2).$$

Obviously x_1, x_2 is an unconditioned weak sequence.

(b) Consider $y_2 T_1 - y_1 T_2 \in Z(K(x_1, x_2; R))$. Then $0 \neq zy_2 T_1 - zy_1 T_2 = zy_2 T_1 \in \mathfrak{m}Z(K)$. Notice that R is a homogeneous ring. Assume now that $zy_2 T_1 \in B(K)$, then there exists a linear form f such that $\delta(fT_1 T_2) = zy_2 T_1$, but $\delta(fT_1 T_2) = fx_1 T_2 - fx_2 T_1$, hence $f \in (0 : x_1) \subseteq \mathfrak{m}^2$ a contradiction.

REMARK 3. The example in the remark to Proposition 1 shows for $r \geq 3$ that the theorem is not valid without the assumption $(x_1, \dots, \hat{x}_i, \dots, x_n) \subseteq (x_1, \dots, x_n)$ for all $1 \leq i \leq n$. In this particular case $H_1(K(x_1)) = 0$ and $x^{r-1}H_1(K(x_1, x_2)) \neq 0$.

Using the theorem and the following Proposition 2 we now would like to give a new proof of the characterization of Buchsbaum rings in terms of the Koszulhomology. This characterization is already known using Suzuki [7] and Shimoda [3]:

The following are equivalent:

(1) R is a Buchsbaum ring,

(2) for all systems of parameters $\mathfrak{x}$: $H_*(\mathfrak{x}, R)$ is a vector space,

(3) for all systems of parameters $\mathfrak{x}$: $H_1(\mathfrak{x}, R)$ is a vectorspace,

(4) for all systems of parameters $\mathfrak{x}$: $\tilde{H}_*(\mathfrak{x}, R)$ is a vectorspace,

(5) for all systems of parameters $\mathfrak{x}$: $\tilde{H}_1(\mathfrak{x}, R)$ is a vectorspace.

($\tilde{H}_i(\mathfrak{x}, M) := \text{coker}(H_i(\mathfrak{x}, (\mathfrak{x})M) \rightarrow H_i(\mathfrak{x}, M))$ for a R -module M .)

(1) $\Rightarrow$ (2) follows from our theorem and Remark 1. Obviously (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5). We only need to show (5) $\Rightarrow$ (1), hence e.g. every system of parameters $\mathfrak{x}$ is a weak sequence. The following proposition which we prove more general for the module case (cf. Introduction) implies the desired result.

PROPOSITION 2. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R -module and $x_1, \dots, x_n$ a sequence of elements in $\mathfrak{m}$ such that $\tilde{H}_1(x_1^{v_1}, \dots, x_n^{v_n}, M)$ is a vectorspace for all multi-indices $(v_1, \dots, v_n)$ with $v_i \geq 1$, then $x_1^{v_1}, \dots, x_n^{v_n}$ is a weak sequence on M for all $(v_1, \dots, v_n)$.*

PROOF. It is only left to show that $\mathfrak{x}$ is a weak sequence. We will do this in two steps: (1) $\mathfrak{m}((x_1, \dots, x_{n-1})M : x_n) \subseteq (x_1, \dots, x_{n-1})M$. (2) If $\tilde{H}_1(x_1^{v_1}, \dots, x_n^{v_n}; M)$ is a vectorspace for all $v_1, \dots, v_n$, then $\tilde{H}_1(x_1^{v_1}, \dots, x_{n-1}^{v_{n-1}}; M)$ is a vectorspace for all $v_1, \dots, v_{n-1}$.

Then descending step by step we obtain the conclusion.

To (1). We have the following commutative diagram with exact rows:

(+)

$$\begin{array}{ccccccc} 0 & \rightarrow & K(x_1, \dots, x_{n-1}; (\mathfrak{x})M) & \rightarrow & K(x_1, \dots, x_n; (\mathfrak{x})M) & \rightarrow & K(x_1, \dots, x_{n-1}; (\mathfrak{x})M)[-1] \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & K(x_1, \dots, x_{n-1}; M) & \rightarrow & K(x_1, \dots, x_n; M) & \rightarrow & K(x_1, \dots, x_{n-1}; M)[-1] \rightarrow 0 \end{array}$$

Taking homology we get the following commutative diagram:

$$\begin{array}{ccccccc} H_1(x_1, \dots, x_n; (\mathfrak{x})M) & \xrightarrow{\varphi_{\mathfrak{x}}} & H_0(x_1, \dots, x_{n-1}; (\mathfrak{x})M) & \rightarrow & \text{coker } \varphi_{\mathfrak{x}} & \rightarrow & 0 \\ \downarrow & & \downarrow \psi & & \downarrow \psi_0 & & \\ H_1(x_1, \dots, x_n; M) & \xrightarrow{\varphi} & H_0(x_1, \dots, x_{n-1}; M) & \rightarrow & \text{coker } \varphi & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \tilde{H}_1(x_1, \dots, x_n; M) & \rightarrow & \text{coker } \psi & \rightarrow & \text{coker } \psi_0 & & \end{array}$$

In other terms

$$\begin{array}{ccccccc} H_1(\mathfrak{x}; (\mathfrak{x})M) & \rightarrow & (\mathfrak{x})M / (\mathfrak{x})(x_1, \dots, x_{n-1})M & \xrightarrow{x_n} & (\mathfrak{x})^2 M / (\mathfrak{x})(x_1, \dots, x_{n-1})M & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_1(\mathfrak{x}; M) & \rightarrow & M / (x_1, \dots, x_{n-1})M & \xrightarrow{x_n} & (\mathfrak{x})M / (x_1, \dots, x_{n-1})M & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \tilde{H}_1(\mathfrak{x}; M) & \rightarrow & M / (\mathfrak{x})M & \xrightarrow{x_n} & (\mathfrak{x})M / (x_1, \dots, x_{n-1}, x_n^2)M & & \end{array}$$

By the snake lemma $\tilde{H}_1(\mathfrak{x}, M)$ maps onto the kernel of

$$M / (\mathfrak{x})M \xrightarrow{x_n} (\mathfrak{x})M / (x_1, \dots, x_{n-1}, x_n^2)M,$$

hence

$$m((x_1, \dots, x_{n-1}, x_n^2)M : x_n) \subseteq (x_1, \dots, x_n)M.$$

Now let $\alpha \in (x_1, \dots, x_{n-1})M : x_n$ we have to show that $\alpha m \subseteq (x_1, \dots, x_{n-1})M$. But

$$\alpha x_n^k \in (x_1, \dots, x_{n-1})M \subseteq (x_1, \dots, x_{n-1}, x_n^{k+1})M \quad \text{for all } k \geq 1,$$

so $\alpha x_n^k \in (x_1, \dots, x_{n-1}, x_n^{2k})M$. Now repeating the above argument for $m\alpha \subseteq (x_1, \dots, x_{n-1}, x_n^k)M$ for all $k \geq 1$, hence $m\alpha \subseteq (x_1, \dots, x_{n-1})M$.

To (2). Again taking homology from (+) we derive Diagram A.

By the snake lemma we have an exact sequence

$$K \rightarrow C(x_1, \dots, x_n; M) \rightarrow \tilde{H}_1(\mathfrak{x}; M).$$

The map $K \rightarrow C(x_1, \dots, x_n; M)$ is actually the zero map. To prove this, consider $\alpha \in M$ with $\bar{\alpha} \in K$, then $\alpha \in (x_1, \dots, x_{n-1})M$. Let $\alpha = \sum_{i=1}^{n-1} x_i \beta_i$, $\beta_i \in M$ then

$$x_n \alpha = \sum_{i=1}^{n-1} x_n x_i \beta_i,$$

so $z = \alpha T_n - \sum_{i=1}^{n-1} \beta_i x_n T_i$ is a cycle in $K(\mathfrak{x}, (\mathfrak{x})M)$ which maps onto $\bar{\alpha}$. But $z = \delta(\sum_{i=1}^{n-1} \beta_i T_i T_n)$ is a boundary in $K_1(x_1, \dots, x_n; M)$. So $C(x_1, \dots, x_n; M)$ is a vector-space since $\tilde{H}_1(\mathfrak{x}; M)$ is.

Again the same argument is true for the sequence $x_1, \dots, x_{n-1}, x_n^k$ for all $k \geq 1$, hence for all $k \geq 1$: $C(x_1, \dots, x_{n-1}, x_n^k; M)$ is a vectorspace. Therefore also

$$\begin{aligned} C'(x_1, \dots, x_{n-1}, x_n^k; M) \\ := \operatorname{coker}(H_1(x_1, \dots, x_{n-1}; (x_1, \dots, x_{n-1}, x_n^k)M) \rightarrow H_1(x_1, \dots, x_{n-1}; M)) \end{aligned}$$

is a vectorspace.

Finally consider Diagram B.

By the snake lemma we have the exact sequence

$$L \rightarrow \tilde{H}_1(x_1, \dots, x_{n-1}; M) \rightarrow C'(x_1, \dots, x_{n-1}, x_n^k; M).$$

So $\tilde{H}_1(x_1, \dots, x_{n-1}; M)$ modulo the submodule generated by all cycles $z = \sum_{i=1}^{n-1} \alpha_i T_i$ with $\alpha_i \in (x_1, \dots, x_{n-1}, x_n^k)M$ is a vectorspace, hence for every cycle $z \in K_1$ we have

$$mz \subseteq \bigcap_{k \geq 1} (x_1, \dots, x_{n-1}, x_n^k)K_1 = (x_1, \dots, x_{n-1})K_1,$$

so $m\tilde{H}_1(x_1, \dots, x_{n-1}; M) = 0$.

We now want to look at the minimal R -free resolution of $R/(\mathfrak{x})$ for an unconditioned weak sequence $\mathfrak{x}$.

By Lemma 2 the Koszulcomplex is always a (module-)direct summand of this resolution for $n \leq 2$. But beside this fact in general very little can be said about what this resolution looks like. If $\mathfrak{x}$ is just a sequence of one element x_1 , then the resolution of $R/(x_1)$ is given by

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \xrightarrow{\varphi} R \xrightarrow{x_1} R/(x_1) \rightarrow 0,$$

where (F_*, d_*) is a direct sum of $\dim_{R/m}(0 : m) = \dim_{R/m} H_1(x_1)$ copies of the minimal R -resolution of the residue class field R/m and φ maps the basiselements of F_0 to representatives of a minimal system of generators of $H_1(x_1)$. So in this case finding the R -resolution of $R/(x_1)$ is equivalent to finding the R -resolution of R/m .

$$\begin{array}{ccccccc}
0 & \rightarrow & H_1(x_1, \dots, x_{n-1}; (\varepsilon)M)/x_n H_1(\dots) & \rightarrow & H_1(\varepsilon; (\varepsilon)M) & \rightarrow & (\varepsilon)(x_1, \dots, x_{n-1})M : x_n / (\varepsilon)(x_1, \dots, x_{n-1})M \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & H_1(x_1, \dots, x_{n-1}; M)/x_n H_1(\dots) & \rightarrow & H_1(\varepsilon; M) & \rightarrow & (x_1, \dots, x_{n-1})M : x_n / (x_1, \dots, x_{n-1})M \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & C(x_1, \dots, x_n; M) & \rightarrow & \tilde{H}_1(\varepsilon; M) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

DIAGRAM A

$$\begin{array}{ccccccc}
H_1(x_1, \dots, x_{n-1}; (x_1, \dots, x_{n-1})M) & \rightarrow & H_1(x_1, \dots, x_{n-1}; (x_1, \dots, x_{n-1}, x_n^k)M) & \rightarrow & L & \rightarrow & 0 \\
\vdots & & \downarrow & & \downarrow & & \\
0 \rightarrow & H_1(x_1, \dots, x_{n-1}; M) & \rightarrow & H_1(x_1, \dots, x_{n-1}; M) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \\
& \tilde{H}_1(x_1, \dots, x_{n-1}; M) & \rightarrow & C'(x_1, \dots, x_{n-1}, x_n^k; M) & &
\end{array}$$

DIAGRAM B

So one may ask in general: Assume we know the R -resolutions of the homology modules $H_i(\mathfrak{x})$, what can be said about the R -resolution of $R/(\mathfrak{x})$ in terms of these given resolutions?

We want to give an answer to this question in a very special case: namely $\mathfrak{x} = x_1, x_2$ is an unconditioned weak sequence of two elements and $\mathfrak{x}$ is unconditioned $\mathfrak{m}$ -relatively regular with respect to $\mathfrak{m}(\mathfrak{x})$.

In this case $H_1(\mathfrak{x})$, $H_2(\mathfrak{x})$ are vectorspaces by Remark 1 to the theorem. Denote M , N free R -modules of rank $\dim H_1(x)$ and $\dim H_2(\mathfrak{x})$ respectively and denote (G, d) the resolution of the residue class field $R/\mathfrak{m}$. Below we will give an equivalent condition for the R -resolution of $R/(\mathfrak{x})$ to be isomorphic to

$$(\times) \quad K \oplus (G \otimes M[-2]) \oplus (G \otimes N[-3])$$

with obvious maps $G_0 \otimes M[-2] \rightarrow K_1$, $G_0 \otimes N[-3] \rightarrow K_2$ killing the homology $H_*(K)$.

Denote $\{z_j | j = 1, \dots, s\}$ a set of cycles in $K_1(\mathfrak{x})$ which represent a $R/\mathfrak{m}$ vector-space basis of $H_1(\mathfrak{x})$. Let $m_1, \dots, m_n$ be a minimal system of generators of the maximal ideal $\mathfrak{m}$. For every pair m_i, z_j we choose an element $\gamma(m_i, z_j) \in \mathfrak{m}$ such that $\delta(\gamma(m_i, z_j)T_1T_2) = m_i z_j$. This is possible, see Remark 1. Let $\mathfrak{A}$ be the R -ideal

$$\mathfrak{A} := \left(\sum_i r_i \gamma(m_i, z_j) / j = 1, \dots, s; \sum_i r_i m_i = 0 \right).$$

Notice that the choices for $\gamma(m_i, z_j)$ are determined up to socle-elements, so as any relation $\sum r_i m_i = 0$ only involves $r_i \in \mathfrak{m}$ the ideal $\mathfrak{A}$ is well defined.

If $\sum r_i m_i = 0$ then

$$\sum_i r_i \gamma(m_i, z_j)(x_1 T_2 - x_2 T_1) = \delta \left(\sum_i r_i \gamma(m_i, z_j) T_1 T_2 \right) = \left(\sum_i r_i m_i \right) z_j = 0.$$

Hence $\sum_i r_i \gamma(m_i, z_j) \in (0 : \mathfrak{m})$, so we obtain $\mathfrak{A} \subseteq (0 : \mathfrak{m})$.

Claim. The minimal R -resolution (F, d) of $R/(\mathfrak{x})$ is isomorphic to $(\times)$ if and only if $\mathfrak{A} = 0$.

PROOF. Clearly the R -resolution F of $R/(\mathfrak{x})$ starts with

$$F_0 = R,$$

$$F_1 = RT_1 \oplus RT_2; T_1 \mapsto x_1, T_2 \mapsto x_2,$$

$$F_2 = RT_1 T_2 \oplus G_0 \otimes M[2].$$

Denote S_j , $j = 1, \dots, s$, a free basis of $M[-2]$ and let $1 \otimes S_j$ map to z_j for $j = 1, \dots, s$.

So we may define for $i = 1, \dots, n$, $j = 1, \dots, s$:

$$d(V_i \otimes S_j) = m_i(1 \otimes S_j) - \gamma(m_i, z_j)T_1 T_2,$$

where $V_1, \dots, V_n$ denote a R -basis of G_1 which is mapped to $m_1, \dots, m_n$ in G .

If $\mathfrak{A} = 0$, then we can define

$$d|_{(G \otimes M[-2])_{\geq 4}} = (d_G)_{i \geq 2} \otimes \text{id}|_{M[-2]},$$

$$d|_{(G \otimes N[-3])_{\geq 4}} = (d_G)_{i \geq 1} \otimes \text{id}|_{N[-3]}$$

and $G_0 \otimes N[-3] \rightarrow K_2$ the obvious map killing the homology $H_2(K)$. So we have the resolution as desired.

If $\mathfrak{A} \neq 0$, then there is a relation $\sum r_i m_i = 0$, $j \in \{1, \dots, s\}$ with $0 \neq \sigma_j = \sum_i r_i \gamma(m_i, z_j) \in (0 : \mathfrak{m})$. Hence the element $\sigma_j T_1 T_2$ is already killed by $d(G_1 \otimes M[-2])$ and therefore must not be killed by a basiselement of $G_0 \otimes N[-3]$. Also $(\sum_i r_i m_i) \cdot (1 \otimes S_j) = 0$, but $(\sum_i r_i V_i) \otimes S_j$ is not a cycle in F although $\sum r_i V_i$ is a cycle in G . So in this case $(\times)$ even is not a complex. Hence the R -resolution of $R/(\mathfrak{x})$ is different from $(\times)$.

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