REMARK ON THE SIMPLICIAL-COSIMPLICIAL TENSOR PRODUCT

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ABSTRACT. We show that the existence of canonical representatives for the elements of the tensor product (coend) of a simplicial and a cosimplicial set depends only on the Eilenberg-Zilber property of the given cosimplicial set. Thus the second condition which is used in [5] for achieving this result is superfluous.

Let $X: \Delta^{\text{op}} \rightarrow S$ be a simplicial set and $Y: \Delta \rightarrow S$ a cosimplicial set. We consider X as a $\mathbb{N}$ -graded set with Δ acting on the right and correspondingly Y is a $\mathbb{N}$ -graded set with Δ acting on the left. The Eilenberg-Zilber Lemma states that every $x \in X$ has a unique decomposition

$$(1) \quad x = x^+ x^\circ$$

with x^+ nondegenerate and x° surjective. We assume that Y has the dual property, i.e.

every $y \in Y$ has a unique decomposition

$$(2) \quad y = y^+ y^\circ$$

with y^+ injective and y° interior.

(That the proof of the Eilenberg-Zilber Lemma fails to be dualizable depends on the fact, that any surjective map is uniquely determined by the set of its sections; but different injective maps with the same one-element domain and the same codomain have the same set of retractions. Thus the Eilenberg-Zilber property for cosimplicial sets is a real restriction; see [2, 4.4 and 4] for a further discussion of this phenomenon.)

Now take

$$(3) \quad T_n = \{(x, y) \mid x \in X, y \in Y, \text{degree } x = \text{degree } y = n\}$$

for every $n \in \mathbb{N}$ and

$$(4) \quad T = \coprod_{n \in \mathbb{N}} T_n;$$

thus T also is a $\mathbb{N}$ -graded set.

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A pair $(x, y) \in T$ is called *similar* to the pair $(x', y') \in T$ if there is an operator $\alpha \in \Delta$ such that

$$(5) \quad x = x'\alpha, \quad \alpha y = y'.$$

Similarity generates an equivalence relation $\sim$ on T such that

$$(6) \quad (x'\alpha, y) \sim (x', \alpha y)$$

for all suitable α . The set of all equivalence classes is called the *tensor product* (coend [3]) of X and Y .

A pair $(x, y) \in T$ is called *minimal*, if x is nondegenerate and y is interior. Our aim is to prove

THEOREM. *Every equivalence class in T contains exactly one minimal element.*

To this end we follow the lines of the proof in [1, 2.1] formulated for “subdivision” functors; see also [5]. We define maps $t_l, t_r, t: T \rightarrow T$ (M_2, M_1, M in [5]) by taking

$$(7) \quad t_r(x, y) = (xy^+, y^\circ),$$

$$(8) \quad t_l(x, y) = (x^+, x^\circ y),$$

$$(9) \quad t = t_l \circ t_r.$$

Then clearly

$$(10) \quad (x, y) \sim t_l(x, y) \sim t_r(x, y) \sim t(x, y),$$

$$(11) \quad t(x, y) \neq (x, y) \Rightarrow \text{degree } t(x, y) < \text{degree } (x, y),$$

$$(12) \quad t_r(x, y) = (x, y) \Leftrightarrow y \text{ interior},$$

$$(13) \quad t_l(x, y) = (x, y) \Leftrightarrow x \text{ nondegenerate},$$

and finally

$$(14) \quad t(x, y) = (x, y) \Leftrightarrow (x, y) \text{ minimal}.$$

Since the set of degrees is bounded below, it follows from (11), that for any pair $(x, y) \in T$ the sequence

$$(15) \quad (t^n(x, y))_{n \in \mathbb{N}}$$

becomes stationary. Thus by (14) it contains a minimal pair, which by (10) is equivalent to the initial pair.

This proves the existence of a minimal pair in every class. The key to the uniqueness lies in the

LEMMA. *If (x, y) is similar to (x', y') then $t_r(x', y')$ is similar to $t(x, y)$.*

This comes out by taking the operator

$$(16) \quad \alpha' = (x'(\alpha y^+)^+)^{\circ}((\alpha y^+)^{\circ} y^{\circ})^+$$

where the exponent $+$ on an operator denotes its injective part while the exponent $^{\circ}$ stands for the surjective component.

Now, for our goal it is enough to show that, if we start with two sequences (15) with similar pairs in T , we end up with the same minimal pair. So let the pair (x, y)

be similar to the pair (x', y') . The lemma provides us with the inductive argument for showing:

For all $n \in \mathbb{N}$ the pair $t, t^n(x, y)$ is similar to the pair $t^n(x', y')$. But analogously as before, we have for sufficiently large n

$$t, t^n(x, y) = t^n(x, y)$$

and both $t^n(x, y)$ and $t^n(x', y')$ are minimal pairs. In view of the equations (5) we see that in this case the corresponding α is injective and surjective, thus $\alpha = \text{id}$, i.e.

$$t^n(x, y) = t^n(x', y')$$

which finishes the proof.

The reason for having written this note is that almost nobody seems to have read the German paper [1].

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