ON THE UNIQUENESS OF GENERALIZED MATRIX FUNCTIONS

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ABSTRACT. We consider the sets of "singular" matrices corresponding to generalized matrix functions. The latter are generalizations of the determinant function. It is shown that two generalized matrix functions determine the same set of "singular" matrices if and only if they differ by a scalar multiple.

Let M_n denote the set of all $n \times n$ matrices over a field F and let S_n be the group of all permutations of $\{1, \dots, n\}$. We denote the identity of M_n by I and the identity of S_n by e . For each mapping $\chi: S_n \rightarrow F$ we define the generalized matrix function $d_\chi: M_n \rightarrow F$ by

$$(1) \quad d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where $A = (a_{ij}) \in M_n$. These functions have been studied extensively when χ is the trivial extension of an irreducible character of a subgroup of S_n .

In particular, if χ is the alternating character, $\text{sgn}(\sigma) = \pm 1$, depending on the parity of $\sigma \in S_n$, then d_χ is the determinant function; or, if $\chi \equiv 1$ then d_χ is the permanent function. Recent interest in the topic of generalized matrix functions was initiated by Marcus and Minc [1] although they appear to have been first introduced by Schur [4].

The singular matrices in M_n are precisely those on which the determinant function vanishes. We study here the relationship between the generalized matrix functions and the corresponding sets of "singular" matrices. We define the *singular set* of the generalized matrix function d_χ by

$$Z(\chi) = \{A \in M_n \mid d_\chi(A) = 0\}$$

and prove that d_χ is essentially determined by $Z(\chi)$. In particular, $Z(\text{sgn})$ is the set of classical singular matrices and clearly $Z(0) = M_n$.

Recently R. Merris [3] has shown that if χ is an irreducible complex character of some subgroup of S_n then the determinant is the only generalized matrix function which is nonzero on every matrix in $Z(\text{sgn})$. Equivalently, for such a character $Z(\chi) \subseteq Z(\text{sgn})$ implies $d_\chi = \det$. The same conclusion is obtained by Marcus [2, p.

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151] but with the hypothesis $Z(\text{sgn}) \subseteq Z(\chi)$. Both of these results are easy consequences of the theorem proved below. Marcus' proof suggests that a more general result could be obtained by use of the Hilbert Nullstellensatz [5, pp. 164–167]. However, we obtain a more general result by a direct argument.

For $\psi: S_n \rightarrow F$ and $\alpha \in S_n$ we define

$$(2) \quad \psi_\eta(\alpha) = \psi(\alpha \circ \eta)$$

for each $\eta \in S_n$. The support of $\chi: S_n \rightarrow F$ is

$$s(\chi) = \{\alpha \in S_n \mid \chi(\alpha) \neq 0\}.$$

The following lemma lists some readily verified facts which we will use in the sequel without explicit mention.

LEMMA. Let $\psi, \chi: S_n \rightarrow F$ be arbitrary mappings and $\eta \in S_n$, then

- (i) $s(\chi) = s(\psi)$ if and only if $s(\chi_\eta) = s(\psi_\eta)$,
- (ii) $a\chi = b\psi$ if and only if $a\chi_\eta = b\psi_\eta$; $a, b \in F$,
- (iii) $P_\eta = (\delta_{i, \eta(j)})$ implies $d_{\chi_\eta}(A) = d_\chi(P'_\eta A)$, $A \in M_n$;
- (iv) $Z(\chi) \subseteq Z(\psi)$ if and only if $Z(\chi_\eta) \subseteq Z(\psi_\eta)$.

THEOREM. Let $\chi, \psi: S_n \rightarrow F$ be any two functions where F is an arbitrary field. Then $Z(\chi) \subset Z(\psi)$ if and only if $\psi = a\chi$ for some $a \in F$. Further, if $a \neq 0$ then $Z(\chi) = Z(\psi)$.

PROOF. Assume $Z(\chi) \subset Z(\psi)$. If $s(\psi)$ is empty then $Z(\psi) = M_n$ and the implication follows with $a = 0$. If $e \notin s(\psi)$ choose η in (2) so that $e \in s(\psi_\eta)$. Thus we assume $e \in s(\psi)$, further, $e \in s(\chi)$. Otherwise, $d_\chi(I) = 0$ and $I \in Z(\chi) \subseteq Z(\psi)$ would imply the contradiction $0 = d_\psi(I) = \psi(e)$. Therefore, there exists $a \in F$ such that $\psi(e) = a\chi(e)$.

If it is false that

$$\psi(\tau) = a\chi(\tau)$$

for all $\tau \in S_n$ then

$$R = \{\tau \in S_n \mid \psi(\tau) \neq a\chi(\tau)\}$$

is nonempty and does not contain e . Choose a $\sigma \in R$ with the fewest number of cycles of length greater than 1.

Let T be the set of disjoint cycles whose product is σ . Since $\sigma \neq e$, T contains a nontrivial cycle. Let L denote a maximal subset of T such that the product of cycles in L lies in $s(\chi)$. Since $e \in s(\chi)$, L is nonempty. Let ρ be the product of all cycles in L and note that $\rho \neq \sigma$.

There is a permutation matrix S and integer k such that

$$SP_\sigma S' = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad \text{and} \quad SP_\rho S' = \begin{bmatrix} I_k & 0 \\ 0 & P_2 \end{bmatrix}$$

where P_1 has all 0's on the main diagonal and P_2 may be empty. Let

$$E_1 = \begin{bmatrix} \chi(\rho) & 0 \\ 0 & I_{n-1} \end{bmatrix}, \quad E_2 = \begin{bmatrix} -\chi(\sigma) & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

and

$$A = E_1 S P_\sigma S' + E_2 S P_\rho S'.$$

Set

$$m = \# \{i \mid \sigma(i) = \rho(i)\},$$

which is the number of rows in P_2 above. Directly from definition (1) we determine that

$$d_\chi(S'AS) = 2^m[\chi(\sigma)\chi(\rho) - \chi(\rho)\chi(\sigma)] = 0.$$

For, if both $a_{1\tau(1)} \cdots a_{n\tau(n)}$ and $\chi(\tau)$ are nonzero then $\tau = \sigma$ or $\tau = \rho$, by the choice of σ and the definition of ρ . Since $Z(\chi) \subseteq Z(\psi)$ we have $d_\psi(S'AS) = 0$ as well and

$$ad_\psi(S'AS) = a\chi(\rho)\psi(\sigma) - a\chi(\sigma)\psi(\rho) = 0.$$

Since ρ has fewer cycles than σ ,

$$\psi(\rho) = a\chi(\rho).$$

Therefore, (4) becomes

$$ad_\psi(S'AS) = \psi(\rho)[\psi(\sigma) - a\chi(\sigma)].$$

Also $\psi(\rho)$ is nonzero since $s(\chi) \subseteq s(\psi)$ and $\rho \in s(\chi)$. Since F is a field we conclude

$$\psi(\sigma) = a\chi(\sigma),$$

contradicting the choice of σ .

For the converse, assume $\psi = a\chi$. Then for $A \in M_n$,

$$\begin{aligned} ad_\chi(A) &= \sum_{\tau \in S_n} a\chi(\tau) \prod_{i=1}^n a_{i\tau(i)} \\ &= \sum_{\tau \in S_n} \psi(\tau) \prod_{i=1}^n a_{i\tau(i)} = d_\psi(A). \end{aligned}$$

If $A \in Z(\chi)$ then $0 = ad_\chi(A) = d_\psi(A)$, and $A \in Z(\psi)$.

The following corollaries contain some immediate but interesting consequences of the above theorem.

COROLLARY 1. *If $\chi, \psi: S_n \rightarrow F$ are any functions and $Z(\chi) \subsetneq Z(\psi)$ then $\psi = 0$ and $Z(\psi) = M_n$.*

COROLLARY 2. *If $\chi: S_n \rightarrow F$ is any function then $d_\chi(A) = 0$ for all noninvertible matrices A if and only if $d_\chi = \det$.*

COROLLARY 3. *If $\chi: S_n \rightarrow F$ is any function then $d_\chi(A) \neq 0$ for all invertible A if and only if $d_\chi = \det$.*

The results of Marcus and Merris mentioned previously are special cases of the last two corollaries.

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