COMPLETENESS AND THE CONTRACTION PRINCIPLE

J. M. BORWEIN¹

ABSTRACT. We prove (something more general than) the result that a convex subset of a Banach space is closed if and only if every contraction of the space leaving the convex set invariant has a fixed point in that subset. This implies that a normed space is complete if and only if every contraction on the space has a fixed point. We also show that these results fail if "convex" is replaced by "Lipschitz-connected" or "starshaped".

In the last few years a metric space variational principle due to Ekeland [3] and an equivalent fixed point theorem due to Caristi have been profitably exploited in a variety of contexts ([2, 4, 12], and the references therein). It is easy to deduce the contraction principle [1] from either of these results [2, 12]; and it is equally apparent from the proof of Ekeland's principle given in [3] that the later result is of "contractive type". This leads naturally to the question as to whether Ekeland's principle may be derived directly from the contraction principle? Moreover, in [5] Kirk showed that Caristi's theorem holds only in a complete metric setting and Sullivan [11] showed the same for Ekeland's theorem. Thus we are led to ask under what circumstances a metric space with the *contraction property* (every metric contraction of the space has a fixed point) is complete? Janos [8] calls such spaces *pseudocomplete*. Many other authors ([5, 8, 9, 10] included) have considered the contraction property in some form or other. Our positive result requires the following definition.

Let us say that a subset C of metric space X endowed with a metric d is *uniformly Lipschitz-connected* if there exists a positive constant L such that given any x^0 and x^1 in C there exists an arc $g: [0, 1] \rightarrow C$ with $g(0) = x^0$ and $g(1) = x^1$ such that

$$(1) \quad d(g(s), g(t)) \leq L |s - t| d(g(0), g(1))$$

for $0 \leq s, t \leq 1$. The prototype of a uniformly Lipschitz-connected set is a convex subset of a normed space. In this case $g(s) := sx^1 + (1-s)x^0$ satisfies (1) with $L := 1$. This is also true in certain translation invariant linear metric spaces. It will be apparent that there are many other types of uniformly Lipschitz-connected sets. These include continuously differentiable finite dimensional surfaces and bi-Lipschitz images of uniformly Lipschitz-connected sets.

Received by the editors January 20, 1982 and, in revised form, April 9, 1982.

1980 *Mathematics Subject Classification*. Primary 47H10, 54H25.

Key words and phrases. Contraction mapping, complete metric space, Ekeland's principle.

¹Research partially supported by NSERC Grant A5116.

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THEOREM 1. *Let C be a uniformly Lipschitz-connected subset of a complete metric space X . Then the following are equivalent.*

- (i) C is closed.
- (ii) Every metric contraction of C has a fixed point in C .
- (iii) Every metric contraction of X which leaves C invariant has a fixed point in C .

PROOF. That (i) implies (ii) is immediate from the contraction principle [6] since C is complete in the subspace metric. Clearly (ii) implies (iii). It remains to show that (iii) implies (i). To this end suppose C is not closed. Choose any $\bar{x}$ in the closure of C which is not in C . Select, as we may, points x_k in C for $k \in \mathbb{N}$ such that

$$(2) \quad d(x_k, \bar{x}) \leq 2^{-(k+2)}.$$

Let $g_k: [0, 1] \rightarrow C$ connect x_k and x_{k+1} in uniform Lipschitz fashion as in (1). Now define $g:]0, \infty[\rightarrow C$ by

$$(3) \quad g(r) := \begin{cases} x_0 & \text{for } 1 < r < \infty, \\ g_k(2^{k+1}r - 1) & \text{for } 2^{-(k+1)} < r \leq 2^{-k}. \end{cases}$$

On each interval of the form $]2^{-k+1}, 2^{-k}]$ we have

$$\begin{aligned} d(g(s), g(t)) &= d(g_k(2^{k+1}s - 1), g_k(2^{k+1}t - 1)) \\ &\leq L |s - t| 2^{k+1} d(g_k(0), g_k(1)) \quad \text{by (1),} \\ &= L |s - t| 2^{k+1} d(x_k, x_{k+1}), \end{aligned}$$

so that

$$(4) \quad d(g(s), g(t)) \leq L |s - t|,$$

on using (2) and the triangle inequality. It follows that g is L -Lipschitz on $]0, \infty[$. Thus, on setting $g(0) := \bar{x}$, g extends to a mapping satisfying (4) on $[0, \infty[$. Now let $h: X \rightarrow [0, \infty[$ be given by

$$(5) \quad h(x) := (1/2L)d(x, \bar{x})$$

and define $T: X \rightarrow X$ by

$$(6) \quad T := g \circ h.$$

It is clear from (4) and (5) that T is a metric contraction (with contraction constant $\frac{1}{2}$). Also

$$\bar{x} = g(0) = g(h(\bar{x})) = T(\bar{x}),$$

so that $\bar{x}$ is the unique fixed point of T in X . Finally observe that, as $\bar{x}$ does not belong to C ,

$$(7) \quad h(C) \subset]0, \infty[$$

and (7) and (3) combine to show $T(C) \subset C$. The mapping T thus violates (iii) and the proof is complete. $\square$

COROLLARY 2. *A uniformly Lipschitz-connected metric space has the contraction property if and only if it is complete.*

PROOF. Consider the space as a uniformly Lipschitz-connected subset of its metric completion [6] and apply Theorem 1. $\square$

In particular a normed space is complete if and only if it has the contraction property. One can not hope to extend Theorem 1 or Corollary 2 much as the following example illustrates.

EXAMPLE 3. *There exists an incomplete nonuniformly Lipschitz-connected subset of the Euclidean plane with the contraction property.*

Let

$$(8) \quad C := \{(x, y) \in \mathbf{R}^2 \mid 0 < x \leq 1, y = \sin(1/x)\}.$$

Since every initial segment of C is smooth and so uniformly Lipschitz-connected only the claim that C has the contraction property needs proof. For $\delta > 0$ define

$$(9) \quad \begin{aligned} C^-(\delta) &:= \{(x, y) \in \mathbf{R}^2 \mid 0 < x < \delta\} \cap C \quad \text{and} \\ C^+(\delta) &:= \{(x, y) \in \mathbf{R}^2 \mid \delta \leq x \leq 1\} \cap C. \end{aligned}$$

Suppose that $T: C \rightarrow C$ is k -contractive for $k < 1$. Pick $\epsilon > 0$ so that

$$(10) \quad k\sqrt{\epsilon^2 + 4} < 2.$$

Let $\alpha := \inf\{x \mid (x, y) \in T(C)\}$. There are two possible cases.

Case (i). $\alpha = 0$. Then as $C = C^-(\epsilon) \cup C^+(\epsilon)$

$$(11) \quad 0 = \min\{\inf\{x \mid (x, y) \in T(C^-(\epsilon))\}, \inf\{x \mid (x, y) \in T(C^+(\epsilon))\}\}.$$

Now $C^+(\epsilon)$ is compact and T is continuous. Thus $T(C^+(\epsilon))$ is a compact subset of C and so $\inf\{x \mid (x, y) \in T(C^+(\epsilon))\} > 0$. It follows that

$$(12) \quad 0 = \inf\{x \mid (x, y) \in T(C^-(\epsilon))\}.$$

Also $T(C^-(\epsilon))$ is arcwise-connected in C . Therefore we may use (12) to pick a, b in $T(C^-(\epsilon))$ with $\|a - b\| \geq 2$. Let $a = T(c)$, $b = T(d)$ with c, d in $C^-(\epsilon)$. Then $\|c - d\| \leq \sqrt{\epsilon^2 + 2^2}$ and

$$(13) \quad 2 \leq \|a - b\| = \|T(c) - T(d)\| \leq k\|c - d\| \leq k\sqrt{\epsilon^2 + 4}$$

which contradicts (10). Thus $\alpha = 0$ is impossible.

Case (ii). $\alpha > 0$. Then

$$(14) \quad T(C^+(\alpha)) \subset T(C) \subset C \cap \{(x, y) \in \mathbf{R}^2 \mid \alpha \leq x \leq 1\} = C^+(\alpha).$$

Since $C^+(\alpha)$ is compact and nonempty, T has a fixed point in $C^+(\alpha)$ and so, a fortiori, in C . Thus C has the contraction property. $\square$

In particular C satisfies (ii) of Theorem 1 and not (i). Note also that C does not have the fixed point property, [10]. The preceding results show that in a normed setting the contraction principle and Ekeland's principle are equivalent but that this fails more generally. It follows that there is no derivation of Ekeland's principle entirely from the contraction principle. In [5] Kirk remarks, without example, that the contraction property holds in some incomplete metric spaces. This follows, of course, from the fact that there are nonclosed subsets of $\mathbf{R}^n$ with the fixed point property [10].

Recall that a subset S of a normed space is said to be *starshaped at s in S* if $\text{conv}\{x, s\} \subset S$ whenever x is in S . Starshaped sets thus provide another natural class of arcwise-connected subsets which are not, in general, Lipschitz-connected. It

might, nonetheless, be conjectured that Theorem 1 remains true with “uniformly Lipschitz-connected” replaces by “starshaped”; after all starshaped sets are well behaved topologically and share many other properties with convex sets. Our final example shows that this conjecture is false.

EXAMPLE 4. *There exists an incomplete starshaped subset of the Euclidean plane with the contraction property.*

Let C be defined by

$$(15) \quad C := \bigcup \{L_k \mid k \in \mathbf{N}\}$$

where

$$(16) \quad L_k := \text{conv}\{(0,0), (1, 2^{-k})\}.$$

Then C is starshaped at zero and is not closed since $\text{conv}\{(0,0), (1,0)\} \subset \text{cl } C$. Let T be a contraction on C . Again there are two cases.

Case (i). $T^n(C)$ does not contain zero for some n in $\mathbf{N}$. Now $T^n(C)$ is a connected subset of C and must therefore lie in L_j for some j in $\mathbf{N}$. It follows that T^n leaves L_j invariant and so has a fixed point in L_j . As T is a contraction it must share this fixed point.

Case (ii). No such n exists. Then

$$(17) \quad (0,0) \in \bigcap \{T^n(C) \mid n \in \mathbf{N}\}.$$

Since T is a contraction on a bounded metric space the diameter of $T^n(C)$ shrinks to zero. It follows that, for any x in C , $T^n(x)$ converges to $(0,0)$. In consequence $(0,0)$ is a fixed point of T .

In either case T has a fixed point and so C has the contraction property. $\square$

We observe that the previous example may be extended to show that

$$(18) \quad C := \bigcup \{C_k \mid k \in \mathbf{N}\}$$

has the contraction property whenever $\{C_k \mid k \in \mathbf{N}\}$ is a family of closed connected subsets of a bounded complete metric space whose intersection is a singleton point of disconnection. It should also be observed that the set of (15) in Example 4 does have the fixed point property as the argument of [10, p. 21] can be used to show.

It seemingly remains open as to whether the multivalued contraction principle [9] characterizes completeness of a metric space. Another converse result related to ours may be found in [7]. As the referee has observed, the examples and theorem relate closely to the question of when the completion of a locally connected metric space is locally connected. A theorem relevant to this is stated in [8].

Thanks are due to my colleagues Charles Coffman, Michael Edelstein, Victor Mizel and Juan Schäffer for several useful conversations on this topic. Thanks are also due to the referee for pointing out certain relevant references.

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DEPARTMENT OF MATHEMATICS, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, B3H 4H8, CANADA
(Current address)