L^p -BOUNDEDNESS OF A CERTAIN CLASS OF MULTIPLIERS ASSOCIATED WITH CURVES ON THE PLANE. II

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ABSTRACT. L^p -boundedness of some multiplier, almost constant along curves, is proved. The interval of p 's depends on the rate of decaying along the above curves.

Introduction. We get positive results for some multiplier considered in [3] after introducing a "weight" factor not heavy enough to allow us to apply Marcinkiewicz's multiplier theorem. The method used is a "geometric decomposition" as used in [1] for Bochner-Riesz multipliers. It can be proved that the present results are sharp in a similar way to that used in [3] for "Schrödinger multiplier". For the sake of simplicity Theorem 2 is stated in a particular case, nevertheless, the method remains valid for a wider range of functions. I would like to thank Professor A. Cordoba for the orientation and advice he has given to me.

1. Let us summarize some version of the theorems in [1] for more general curves than the circumference.

Let $\gamma(t)$ be an arc of a smooth curve in $\mathbf{R}^2$, $t \in [t_0, t_1]$, such that its curvature is positive at any $t \in [t_0, t_1]$.

Pick a $\delta > 0$ small enough for there to be a collar or strip around γ of width δ such that the mapping,

$$(x_1, x_2) \in \text{Collar}_\delta \gamma \rightarrow (s, t)$$

where $(x_1, x_2) = x = \gamma(t) + s\eta(\gamma(t))$ and η denotes the normal of r at $\gamma(t)$, is one-to-one in the collar and has derivatives of a suitable order.

Let ϕ and Ψ be C_0^∞ functions s.t. $\phi(0) = 1$; ϕ is supported on $(-1, 1)$ and Ψ is supported in $[t_0, t_1]$. Notice the function defined by

$$m(x) = \begin{cases} \phi\left(\frac{s}{\delta}\right)\Psi(t) & \text{when } x = \gamma(t) + s\eta(\gamma(t)), x \in \text{Collar}_\delta \gamma, \\ 0 & \text{otherwise} \end{cases}$$

is in C_0^∞ .

Define the operator

$$(T_\delta f)^\wedge(\xi) = m(\xi)\hat{f}(\xi), \quad f \in C_0^\infty(\mathbf{R}^2).$$

Let $\kappa(t)$ be the curvature of γ at $\gamma(t)$; consider $\kappa_2 \leq \kappa(t) \leq \kappa_1$ for $t \in [t_0, t_1]$ and $\kappa_2 > 0$, then the following holds.

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THEOREM 1.

$$\|T_\delta f\|_4 \leq C \frac{\kappa_1}{\kappa_2} |1 + \log \delta \kappa_1|^{1/4} \|f\|_4 \quad \text{for every } f \in C_0^\infty(\mathbf{R}^2).$$

We refer to the proof given in [1] for the case of γ a circumference. For further details see [2].

LEMMA 2.

$$\|T_\delta f\|_\infty \leq CL\kappa_1^{1/2}\delta^{-1/2} \|f\|_\infty$$

for any $f \in L^\infty(\mathbf{R}^2)$, where L is the length of the curve in $t \in [t_0, t_1]$.

PROOF. It suffices to see that

$$\|\hat{m}(\cdot)\|_{L^1} \leq CL\kappa_1^{1/2}\delta^{-1/2}.$$

It can be reduced by dilations to the case m_1 when $\kappa_1 = 1$, length $L' = L\kappa_1$ and $\delta' = \delta\kappa_1$, for $\|\hat{m}_1\|_{L^1} = \|\hat{m}\|_{L^1}$,

$$\|\hat{m}(\cdot)\|_1 \leq CL'\delta'^{-1/2} = CL\kappa_1\delta^{-1/2}\kappa_1^{-1/2}.$$

Following the theorem in [1] let us decompose the arc of the curve into $L\delta^{-1/2}$ pieces such that

(a) the length of each of them is $\delta^{1/2}$;

(b) the angle between the normals at two points in the same piece is less than $\delta^{1/2}$.

According to this partition of the curve we are going to break down the multiplier; take $\{\Psi_j\}_{j=1, \dots, L\delta^{-1/2}}$ to be a smooth partition of the unity in $[0, L]$ constructed in the following way:

$\Psi_1(t) = \Psi_0(\delta^{-1/2}t)$ where $\Psi \in C_0^\infty(\mathbf{R})$ is supported in $[-1, 1]$ and its Schwarz's bounds are independent of δ ,

$$\begin{aligned} \Psi_j(t) &= \Psi_1(t - j\delta^{1/2}), \\ \sum_{j=1}^{L\delta^{-1/2}} \Psi_j(t) &\quad \text{for } t \in [0, L]. \end{aligned}$$

We can suppose the curve is referred to arclength t . Then $m(\xi) = \phi(s/\delta)\Psi(t) = \sum_1^{L\delta^{-1/2}} \phi(s/\delta)\Psi_j(t)\Psi(t)$ and for saving notation let us call on $\Psi_j(t)\Psi(t) = \Psi_j(t)$; $m_j(\xi) = \phi(s/\delta)\Psi_j(t)$.

By (a) and (b), $m_j(\xi)$ is supported in a rectangle of length $2\delta^{1/2}$ and width 2δ , whose direction is the tangent to the curve at $t = j\delta^{1/2}$. Set $\widehat{T_j f}(\xi) = m_j(\xi)\hat{f}(\xi)$. Then

$$Tf(x) = \sum_{j=1}^{L\delta^{-1/2}} T_j f(x).$$

Now, $\widehat{m_j}(x) = e^{2\pi i x_j \cdot x} K_j(\rho_j x)$, where x_j is the middle point of the j -piece of curve, ρ_j is a rotation and K_j satisfies

$$(1) |K_j(x_1, x_2)| \leq C_\rho \delta^{3/2} |x_2 \delta^{1/2}|^p \quad \text{for } p = 0, 1, 2, 3,$$

$$(2) |K_j(x_1, x_2)| \leq C_\rho \delta^{3/2} |x_1 \delta|^{-q} \quad \text{for } q = 0, 1, 2, 3,$$

where C_ρ and C_q are independent of j .

(This is proved using integration by parts and properties on the support of m_j ; see [1] in the case of the circumference.)

Then $\hat{m}(x) = \sum e^{2\pi i x_j x} K_j(\rho_j x)$ and

$$\int |\hat{m}(x)| dx \leq \sum_{j=1}^{L\delta^{-1/2}} \int |K_j(\rho_j x)| dx \leq CL\delta^{-1/2}.$$

The last inequality follows from (1) and (2) by taking a suitable decomposition of $\int_{\mathbb{R}^2} |K_j(x_1, x_2)| dx$.

2. The case of a multiplier along a curve. Let $\phi \in C_0^\infty$ be supported on $[-1, 1]$. Given $y = \gamma(x)$ an increasing function with $\gamma''(x) > 0$, and Ψ a function in $C_0^\infty(\mathbb{R})$ supported in $[a, b]$, $a > 0$, let us consider

$$m(\xi_1, \xi_2) = \phi(\xi_2 - \gamma(\xi_1))\Psi(\xi_1).$$

Then

THEOREM 1'.

$$\|Tf\|_4 \leq C |\log \gamma_1'^{-1} \cdot \kappa_1|^{1/4} \|f\|_4$$

where $(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$, κ_1 is an upper bound of the curvature of $\gamma(x) = y$, γ_1' is a lower bound of $\gamma'(x)$ when $x \in [a, b]$, $c = C_1\kappa_1/\kappa_2$ is the ratio between the maximum and minimum curvatures on $[a, b]$ and c_1 depends only on Schwarz's bounds of ϕ and Ψ .

LEMMA 2'.

$$\|Tf\|_\infty \leq CL\gamma_1'^{-1/2}\kappa_1^{1/2} \|f\|_\infty$$

where L is the length of the arc of the curve $y = \gamma(x)$, $x \in [a, b]$ and C depends only on Schwarz's bounds of ϕ and Ψ .

The proofs are as in Theorem 1 and Lemma 2; note that $\gamma_1'^{-1}$ plays the role of δ there.

THEOREM 2. Let

$$m(\xi_1, \xi_2) = \frac{1}{1 + (\xi_2 - \xi_1^n)^2} \phi(\xi_1, \xi_2)$$

be a bounded function, where n is an integer $n \geq 2$ and $\phi(\xi_1, \xi_2)$ a C^∞ function such that

$$\phi(\xi_1, \xi_2) = \frac{1}{|(\xi_1, \xi_2)|^2} \quad \text{when } |(\xi_1, \xi_2)| > N, \quad \frac{1}{2} \geq \alpha > 0.$$

Then m is a multiplier on L^p for

$$\frac{4}{3 + 2\alpha} < p < \frac{4}{1 - 2\alpha}.$$

PROOF. Let us take a partition of unity on $(0, \infty)$, $\{\Psi_j\}_{j=0}^\infty$ such that $\Psi_j(x) = \Psi_1((x - 2^j)/2^{j-1})$ and $\text{supp } \Psi_j \subset [0, 6]$. (We can do the same for $x \leq 0$.)

Let us split down

$$m(\xi_1, \xi_2) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} m(\xi_1, \xi_2) \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1).$$

Set

$$m_{l,k} = m(\xi_1, \xi_2) \cdot \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1).$$

Except for a finite number of l 's and k 's depending only on N ,

$$m_{l,k}(\xi_1, \xi_2) = \frac{1}{1 + (\xi_2 - \xi_1^n)^2} \cdot \frac{1}{(\xi_1^2 + \xi_2^2)^{\alpha/2}} \Psi_1\left(\frac{\xi_2 - \xi_1^n - 2^l}{2^{l-1}}\right) \Psi_1\left(\frac{\xi_1 - 2^k}{2^{k-1}}\right).$$

After a change of variables, taking the new variables $(\xi_2 - 2^l)/2^{l-1}$, $\xi_1/2^{(l-1)/n}$, the norm of $m_{l,k}$ as a multiplier on L^p is the same as the norm of

$$(2.1) \quad \bar{m}_{l,k}(\xi_1, \xi_2) = \frac{1}{2^{l(\alpha+2)}} \frac{1}{2^{-2l} + (\xi_2 - \xi_1^n)^2 + 4(\xi_2 - \xi_1^n) + 4} \\ \cdot \Psi_1(\xi_2 - \xi_1^n) \Psi_1(\xi_1 2^{l/n} - 2) \frac{1}{(\xi_1^2 2^{2l/n-2l} + \xi_2^2 + 4\xi_2 + 4)^{\alpha/2}}$$

which is the product in the case $kn > l$ of

$$(2.2) \quad \bar{m}_{l,k}^1(\xi_1, \xi_2) = \frac{1}{2^{l(\alpha+2)} |2^{-2l} + (\xi_2 - \xi_1^n)^2 + 4(\xi_2 - \xi_1^n) + 4|} \\ \cdot \Psi_1(\xi_2 - \xi_1^n) \Psi_1(\xi_1 2^{l/n-k} - 2)$$

and

$$(2.3) \quad \bar{m}_{l,k}^2 = \chi_{[2^{kn-l+n}, 2^{kn-l+2n+6}]}(\xi_2) \chi_{[2^{k-l/n+1}, 2^{k-l/n+3}]}(\xi_1) \\ \cdot \frac{1}{(\xi_1^2 2^{2l/n-2l} + \xi_2^2 + 4\xi_2 + 4)^{\alpha/2}}.$$

Looking at (2.2) we notice that the function whose argument is $(\xi_2 - \xi_1^n)$ has Schwarz's norms bounded independently of l and k , and the support of $\Psi_1(\xi, 2^{l/n-k} - 2)$ is contained in $[2^{k-l/n-1}, 2^{k-l/n-2}]$. Then it is possible to apply Theorem 1' and to bound its (L^4, L^4) -norm by

$$\frac{C}{2^{l(\alpha+2)}} (\log |\gamma'_1 k|)^{1/4} \leq \frac{C}{2^{l(\alpha+2)}} k^{1/4}.$$

In order to study $\bar{m}_{l,k}^2$, let us take $\phi_{k,l}$ and $\Omega_{k,l}$ in $C_0^\infty(\mathbf{R})$, such that

$$\text{supp } \phi_{k,l} \subset [2^{kn-l+n-1}, 2^{kn-l+3n+2}]$$

and $\phi_{k,l} \equiv 1$ on $[2^{kn-l+n}, 2^{kn-l+3n+1}]$; $\text{supp } \Omega_{k,l} \subset [2^{k-l/n}, 2^{k-l/n+4}]$ and $\phi_{k,l} \equiv 1$ on $[2^{k-l/n+1}, 2^{k-l/n+3}]$. Namely, $\phi_{k,l}(x) = \phi(x/2^{kn-l+n})$ where $\text{supp } \phi_0 \subset [\frac{1}{2}, 2^{3n} + 3]$, then $\bar{m}_{l,k}^2$ is the product of

$$\chi_{[2^{kn-l+n}, 2^{kn-l+3n+1}]}(\xi_2) \cdot \chi_{[2^{k-l/n+1}, 2^{k-l/n+3}]}(\xi_1)$$

and

$$\phi_{k,l}(\xi_2) \cdot \Omega_{k,l}(\xi_1) \frac{1}{|\xi_1^2 \cdot 2^{2l/n-2l} + \xi_2^2 + 4\xi_2 + 4|^{\alpha/2}}.$$

The first term of the product corresponds to a (p, p) operator with constant norm independent of k and l . The second one, after changing the variable ξ_2 by $\xi_2/2^{kn-l+n}$, has the same norm as

$$(2.4) \quad \frac{1}{2^{(kn-l+n)\alpha}} \phi_0(\xi_2) \Omega_{k,l}(\xi_1) \cdot \frac{1}{|\xi_1^2 / (2^{2kn-2l/n} + 2n) + \xi_2^2 + 4\xi_2/2^{kn-l+n} + 4/2^{2kn-2l-2n}|^{\alpha/2}}$$

which, as operator $L^p \rightarrow L^p$, $1 < p \leq \infty$, is bounded by $c/2^{(kn-l+n)\alpha}$. Therefore,

$$(2.5) \quad \|m_{l,k}\|_{L^4, L^4} \leq \frac{c}{2^{kn\alpha} \cdot 2^{2l}} \cdot k^{1/4}.$$

In the case $kn \leq 1$, we decompose (2.1) in the product of

$$m_{l,k}^1(\xi_1, \xi_2) = \frac{1}{2^{l(\alpha+2)}} \frac{\Psi_1(\xi_2 - \xi_1^n) \xi_{[0,2^{3n}]}(\xi_1)}{2^{-2l} + (\xi_2 - \xi_1^n)^2 + 4(\xi_2 - \xi_1^n) + 4}$$

and

$$m_{l,k}^2(\xi_1, \xi_2) = \Psi_1(\xi_1 2^{l/n-k} - 2) \frac{\chi_{[0,2^{3n}]}(\xi_2)}{(\xi_1^2 2^{2l/n-2l} + \xi_2^2 + 4\xi_2 + 4)^{\alpha/2}}.$$

The first one has (p, p) -norm, $1 < p < \infty$, as $1/2^{l(\alpha+2)}$; the second one has the same (p, p) -norm as the multiplier

$$(2.6) \quad \Psi(\xi_1) \frac{1}{2^{(k-l)\alpha}} \frac{\chi_{[0,2^{3n}]}(\xi_2)}{|\xi_1^2 + \xi_2 + \xi_2^2/2^{2k-2l} + 4\xi_2/2^{2k-2l} + 4/2^{2k-2l} + 4|^{\alpha/2}}$$

obtained by the change $\xi_1 \rightarrow (\xi_1 - 2^{k-l/n+1})/2^{k-l/n}$. The Schwarz's norm in (2.6), divided by $1/2^{(k-l)\alpha}$, is independent of k and l and we can see, in the way we did in (2.4) above, that $\|m_{l,k}^2\|_{p,p} \leq c/2^{(k-l)\alpha}$ and then $\|m_{l,k}\|_{p,p} \leq c/(2^{k\alpha} 2^{2l})$ in the case $kn \leq l$. Therefore the sum $\sum_{k,l, kn \leq l} \|m_{l,k}\|_{p,p}$ is convergent and it suffices to consider $m_{l,k}$ with $kn > l$. By Lemma 2',

$$(2.7) \quad \|\bar{m}_{l,k}^1\|_{\infty, \infty} \rightarrow \frac{1}{2^{l(\alpha+2)}} 2^{nk} \cdot 2^{-(k+1)(n-1/2)} \cdot 2^{k(n-1)/2} \\ \sim \frac{c}{2^{l(\alpha+1)}} 2^{kn/2}.$$

So, since (2.4) and (2.7) hold, we deduce that

$$(2.8) \quad \|m_{l,k}\|_{\infty, \infty} \leq \frac{c}{2^{2l} 2^{kn\alpha}} 2^{kn/2}.$$

Interpolation between (L^∞, L^∞) boundedness arising from (2.8) and (L^4, L^4) boundedness from (2.5) implies that

$$\|m_{l,k}\|_{L^p, L^p} \leq \frac{c \cdot k^{1/4}}{2^{2l} 2^{kn\alpha - kn/n + kn/p}}$$

converges if $p < 4/(1 - 2\alpha)$, and the result follows by duality and interpolation.

Note. The method of Theorem 2 could be applied to some other operators, as we noticed; one of them is

$$m(\xi_1, \xi_2) = \frac{1}{(\xi_2 - \xi_1^n)^2 + 1} \frac{1}{(|\xi_1|^2 + 1)^{\alpha/2}}.$$

It is possible to prove that $m(\xi_1, \xi_2)$ is a multiplier on L^p , when

$$\frac{4}{3 + 2^{\alpha/n}} < p < \frac{4}{1 - 2^{\alpha/n}}.$$

Following the method of [3] we could prove that the above mentioned intervals are the best possible; hence, if we set

$$m(\xi_1, \xi_2) = \frac{1}{(\xi_2 - \xi_1^n)^2 + 1} \frac{1}{(|\xi_1|^2 + 1)},$$

$m(\xi_1, \xi_2)$ is a multiplier on L^p in an interval of p 's, neither (2.2) nor $(1, \infty)$ taking n large enough. This multiplier arises from a partial differential equation on the plane.

REFERENCES

1. A. Cordoba, *A note on Bochner-Riesz operators*, Duke Math. J. **46** (1979).
2. A. Ruiz, *Multiplicadores asociados a curvas en el plano y teoremas de restriction de la transformada de Fourier a curvas en $\mathbb{R}^2$ y $\mathbb{R}^3$* , Tesis doctoral, Universidad Complutense de Madrid, 1980.
3. ———, *L^p -boundedness of a certain class of multipliers associated with curves on the plane. I*, Proc. Amer. Math. Soc. (to appear).

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