A RADON-NIKODYM THEOREM FOR NATURAL CONES ASSOCIATED WITH VON NEUMANN ALGEBRAS. II

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ABSTRACT. A natural cone associated with a (σ -finite) von Neumann algebra is considered. Let ξ_0 be a cyclic and separating vector in the cone. For each vector ξ in the cone, there always exists a positive selfadjoint operator t affiliated with the algebra satisfying $\xi = tJ\xi_0$. Certain uniqueness results on t for a given ξ are also obtained.

1. Introduction. We consider a (σ -finite) von Neumann algebra $\mathfrak{M}$ acting standardly on a Hilbert space $\mathcal{H}$. Let $\mathcal{P}^h$ be a natural cone in $\mathcal{H}$, and J be the corresponding modular conjugation, that is, the quadruple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P}^h)$ is a standard form, [2, 3, 4]. We fix a cyclic and separating vector ξ_0 in $\mathcal{P}^h$ so that $\mathcal{P}^h$ may be expressed as the closure of all vectors $xJxJ\xi_0$, $x \in \mathfrak{M}$.

Improving the main result in [5], we will prove

THEOREM. *For each vector ξ in $\mathcal{P}^h$, there always exists a positive selfadjoint operator t affiliated with $\mathfrak{M}$ satisfying $\xi_0 \in \mathcal{D}(t)$, $Jt\xi_0 = JtJ\xi_0 \in \mathcal{D}(t)$, and $\xi = tJtJ\xi_0$.*

In §§4 and 5, we will also prove that (i) the above t is not uniquely determined by ξ as an operator (§5), (ii) however, it is unique in a certain sense (Lemma 5, Proposition 6).

We will use certain terminologies and properties established in [5], which are summarized in the next section. Our standard references on the modular theory and unbounded operators are [9 and 7] respectively.

2. Preliminaries [5]. The quadruple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P}^h)$ and a distinguished cyclic and separating vector ξ_0 in $\mathcal{P}^h$ with the vector functional $\phi_0 = \omega_{\xi_0} \in \mathfrak{M}_*^+$ will be fixed throughout.

For each $\chi \in \mathfrak{M}_*^+$, its unique implementing vector in $\mathcal{P}^h$ is denoted by ξ_χ . (Thus, $\chi = \omega_{\xi_\chi}$ and $\xi_{\phi_0} = \xi_0$.) The operator $x\xi_0 \in \mathfrak{M}\xi_0 \rightarrow Jx^*\xi_\chi \in \mathfrak{M}'\xi_\chi$ is positive and essentially selfadjoint, and its closure is denoted by $\Delta_{\chi\phi_0}^{1/2}$. When $\chi = \phi_0$, we write

$$\Delta = \Delta_{\phi_0} = \Delta_{\phi_0\phi_0}, \text{ the usual modular operator} \quad [9].$$

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We denote the set of all entire elements in $\mathfrak{M}$ (resp. $\mathfrak{M}'$) for the modular automorphism group $\sigma_t = \text{Ad } \Delta^t$ (resp. $\sigma'_t = \text{Ad } \Delta^{-t}$), $t \in \mathbb{R}$, simply by $\mathfrak{M}_0$ (resp. $\mathfrak{M}'_0$).

In what follows, we will also fix an (arbitrary) vector ξ in $\mathfrak{P}^{\natural}$, and set $\phi = \omega_{\xi} \in \mathfrak{M}_*^+$. Hence, we will sometimes write ξ_{ϕ} instead of ξ . In [5], we set

$$(1) \quad \psi(x) = (\Delta^{1/4} x \xi_0 | \xi_{\phi}), \quad x \in \mathfrak{M},$$

and proved $\psi \in \mathfrak{M}_*^+$ and

LEMMA 1 ([5, LEMMA 3.3]). *The unique implementing vector ξ_{ψ} for ψ in $\mathfrak{P}^{\natural}$ belongs to $\mathfrak{D}(\Delta^{1/4}) \cap \mathfrak{D}(\Delta^{-1/4})$.*

As usual, we denote the closure of the positive cone $\mathfrak{M}_+ \xi_0$ (resp. $\mathfrak{M}'_+ \xi_0$) in $\mathfrak{K}$ by $\mathfrak{P}^{\#} = \mathfrak{P}_{\phi_0}^{\#}$ (resp. $\mathfrak{P}^{\flat} = \mathfrak{P}_{\phi_0}^{\flat}$) [9]. It is easy to check

$$(2) \quad \mathfrak{P}^{\#} = \{\zeta \in \mathfrak{K}; (\zeta | \eta) \geq 0 \text{ for all } \eta \in \mathfrak{P}^{\flat}\} = J\mathfrak{P}^{\flat},$$

$$(3) \quad J\zeta = \Delta^{1/2}\zeta \text{ (resp. } J\zeta = \zeta) \text{ if } \zeta \in \mathfrak{P}^{\#} \text{ (resp. } \zeta \in \mathfrak{P}^{\natural}),$$

$$(4) \quad \mathfrak{P}^{\#} \subseteq \Delta^{-1/4}(\mathfrak{P}^{\natural} \cap \mathfrak{D}(\Delta^{-1/4})), \Delta^{1/4}\mathfrak{P}^{\#} \subseteq \mathfrak{P}^{\natural}.$$

(See [2, 9] for details.)

We will frequently use the following result, which is an easy consequence of the spectral decomposition theorem:

LEMMA 2. *Any core for a positive selfadjoint operator h is also a core for h^{α} , $0 < \alpha < 1$.*

3. Proof of the Theorem. In this section, we will prove the Theorem stated in §1, after preparing two lemmas.

Based on Lemma 1, we set

$$\zeta_{\phi} = \Delta^{-1/4}\xi_{\psi}.$$

This is a vector in $\mathfrak{P}^{\#}$ (4). Let a_{ϕ} be a densely defined operator on $\mathfrak{K}$ determined by

$$\mathfrak{D}(a_{\phi}) = \mathfrak{M}'\xi_0, \quad a_{\phi}(x'\xi_0) = x'\zeta_{\phi}, \quad x' \in \mathfrak{M}'.$$

Obviously, a_{ϕ} is affiliated with $\mathfrak{M}$, and due to (2) it is a positive (symmetric) operator (so that it is closable).

LEMMA 3. *The vector $J\zeta_{\phi}$ ($\in \mathfrak{P}^{\flat}$, (2)) belongs to $\mathfrak{D}(a_{\phi}^*)$, and $a_{\phi}^*J\zeta_{\phi} = \xi_{\phi}$.*

PROOF. It suffices to show that

$$(J\zeta_{\phi} | a_{\phi}x'\xi_0) = (\xi_{\phi} | x'\xi_0), \quad x'\xi_0 \in \mathfrak{M}'\xi_0 = \mathfrak{D}(a_{\phi}).$$

However, since $\mathfrak{M}'_0$ is dense in $\mathfrak{M}'$ with respect to the strong operator topology, it suffices to show the above equality for any $x' \in \mathfrak{M}'_0$. For such an x' , we compute

$$\begin{aligned}
(J\zeta_\phi | a_\phi x' \xi_0) &= (J\Delta^{-1/4} \xi_\psi | x' \zeta_\phi) = (J\Delta^{-1/4} \xi_\psi | x' \Delta^{-1/4} \xi_\psi) \\
&= (J\Delta^{-1/4} \xi_\psi | x' \Delta^{-1/4} J \xi_\psi) \quad (\xi_\psi \in \mathfrak{D}^q \text{ and (3)}) \\
&= (J\Delta^{-1/4} \xi_\psi | x' J \Delta^{1/4} \xi_\psi) \quad (J \Delta^{1/4} J = \Delta^{-1/4}) \\
&= (J x' J \Delta^{1/4} \xi_\psi | \Delta^{-1/4} \xi_\psi) \\
&= (\sigma_{i/4}(J x' J) \xi_\psi | \xi_\psi) \\
&= \psi(\sigma_{i/4}(J x' J)) \\
&= (\Delta^{1/4} \sigma_{i/4}(J x' J) \xi_0 | \xi_\phi) \quad ((1)) \\
&= (J x' J \xi_0 | \xi_\phi) \\
&= (\xi_\phi | x' \xi_0) \quad ((3)). \quad (\text{Q.E.D.})
\end{aligned}$$

LEMMA 4. The vector $J\zeta_\phi$ belongs to the domain of the closure $\bar{a}_\phi$, and $\bar{a}_\phi J\zeta_\phi = \xi_\phi$.

PROOF. The previous lemma and (3) yield

$$(5) \quad \xi_\phi = J\xi_\phi = J a_\phi^* J \zeta_\phi.$$

For each $n = 1, 2, \dots$, let x'_n be $J a_\phi^* J e'_n$, where e'_n is the spectral projection of $|J a_\phi^* J| = J(\bar{a}_\phi a_\phi^*)^{1/2} J$ corresponding to the interval $[0, n]$. Thus, $\{x'_n\}_{n=1,2,\dots}$ is a sequence in $\mathfrak{M}'$.

Because of (5), one obtains

$$\lim_{n \rightarrow \infty} \|x'_n \zeta_\phi - \xi_\phi\| = \lim_{n \rightarrow \infty} \|x'_n \zeta_\phi - J a_\phi^* J \zeta_\phi\| = 0.$$

Also, since $J\xi_0 = \xi_0 \in \mathfrak{D}(a_\phi)$ and $a_\phi \subseteq a_\phi^*$,

$$J a_\phi^* J \xi_0 = J a_\phi \xi_0 = J \zeta_\phi.$$

Thus, one obtains

$$\lim_{n \rightarrow \infty} \|x'_n \xi_0 - J \zeta_\phi\| = \lim_{n \rightarrow \infty} \|x'_n \xi_0 - J a_\phi^* J \xi_0\| = 0.$$

The above two convergences mean that the pair $(J\zeta_\phi, \xi_\phi)$ is in the closure of the graph $\Gamma(a_\phi) (\subseteq \mathcal{H} \oplus \mathcal{H})$. (Q.E.D.)

PROOF OF THE THEOREM. Since the closed operator $\bar{a}_\phi$ is positive, it admits a positive selfadjoint extension t (see [1 or 8]). According to [8], t is automatically affiliated with $\mathfrak{M}$, since a_ϕ and $\bar{a}_\phi$ are. Because $a_\phi \subseteq \bar{a}_\phi \subseteq t$ and $J\zeta_\phi = J a_\phi \xi_0 = J t \xi_0$, the Theorem follows from Lemma 4. (Q.E.D.)

4. Uniqueness. We will obtain certain uniqueness results on an operator t appearing in the Theorem.

LEMMA 5. A vector $\zeta_\phi = t\xi_0$ ($\in \mathfrak{D}^\#$) is uniquely determined by a given $\xi = \xi_\phi$ in $\mathfrak{D}^q$ if $\xi_\phi = t J t J \xi_0$ as in the Theorem.

PROOF. When $\xi_\phi = \iota J \iota J \xi_0$ as in the Theorem, for each $x \in \mathfrak{M}_0$, we compute

$$\begin{aligned} \psi(x) &= (\Delta^{1/4} x \xi_0 | \iota J \iota J \xi_0) \quad (\text{recall (1)}) \\ &= (\sigma_{-i/4}(x) \xi_0 | \iota J \iota J \xi_0) \\ &= (\sigma_{-i/4}(x) J t \xi_0 | t \xi_0) \\ &= (x \Delta^{-1/4} J t \xi_0 | \Delta^{1/4} t \xi_0) \\ &= (x \Delta^{1/4} t \xi_0 | \Delta^{1/4} t \xi_0) \quad (t \xi_0 \in \mathfrak{P}^\# \text{ and (3)}). \end{aligned}$$

Thus, $\Delta^{1/4} t \xi_0 \in \mathfrak{P}^\#$ ((4)) is the unique implementing vector ξ_ϕ . Since $\Delta^{1/4}$ is nonsingular, the vector $\xi_\phi = \Delta^{-1/4} \xi_\psi = t \xi_0$ is uniquely determined by ξ_ϕ . (Q.E.D.)

PROPOSITION 6. *Let t be a positive selfadjoint operator affiliated with $\mathfrak{M}$ satisfying $\xi_\phi = \iota J \iota J \xi_0 \in \mathfrak{P}^\#$ (and $\xi_0, J t \xi_0 \in \mathfrak{D}(t)$). Then the (algebraic) product $\Delta^{1/4} t$ is densely defined and closable, and as an operator the closure $(\Delta^{1/4} t)^-$ is uniquely determined by ξ_ϕ . (Furthermore, $|(\Delta^{1/4} t)^-|$ is exactly $\Delta_{\phi\phi_0}^{1/4}$.)*

PROOF. The first assertion is obvious due to

$$\mathfrak{M}'_0 \xi_0 \subseteq \mathfrak{D}(\Delta^{1/4} t), \quad \mathfrak{M}'_0 \xi_0 \subseteq \mathfrak{D}(t \Delta^{1/4}) \subseteq \mathfrak{D}((\Delta^{1/4} t)^*).$$

For the second assertion, we at first note that

$$(\Delta^{1/4} t)^* (\Delta^{1/4} t)^- \supseteq t \Delta^{1/4} (\Delta^{1/4} t)^- \supseteq t \Delta^{1/2} t.$$

For each $x' \in \mathfrak{M}'_0$, we compute

$$\begin{aligned} t \Delta^{1/2} t x' \xi_0 &= t \Delta^{1/2} x' t \xi_0 \\ &= t \sigma'_{i/2}(x') J t \xi_0 \quad (t \xi_0 \in \mathfrak{P}^\# \text{ and (3)}) \\ &= \sigma'_{i/2}(x') \iota J \iota J \xi_0 \\ &= \sigma'_{i/2}(x') \xi_\phi = J (J \sigma'_{i/2}(x') J) \xi_\phi \\ &= \Delta_{\phi\phi_0}^{1/2} (J \sigma'_{i/2}(x') J)^* \xi_0 = \Delta_{\phi\phi_0}^{1/2} J \sigma'_{-i/2}(x'^*) J \xi_0 \\ &= \Delta_{\phi\phi_0}^{1/2} J \Delta^{-1/2} x'^* \xi_0 = \Delta_{\phi\phi_0}^{1/2} x' \xi_0. \end{aligned}$$

Here, we used $J \Delta^{-1/2} x'^* \xi_0 = x' \xi_0$ (see [9]). We thus proved that

$$(\Delta^{1/4} t)^* (\Delta^{1/4} t)^- \supseteq \Delta_{\phi\phi_0}^{1/2} |_{\mathfrak{M}'_0 \xi_0}.$$

We recall that $\mathfrak{M} \xi_0$ is a core for $\Delta_{\phi\phi_0}^{1/2}$. Based on the fact that $\mathfrak{M}_0$ is dense in $\mathfrak{M}$ with respect to the strong*-operator topology, the definition of $\Delta_{\phi\phi_0}^{1/2}$ (in §2) immediately implies that $\mathfrak{M}_0 \xi_0 (= \mathfrak{M}'_0 \xi_0)$ is also a core for $\Delta_{\phi\phi_0}^{1/2}$. Since $(\Delta^{1/4} t)^* (\Delta^{1/4} t)^-$ is always (positive) selfadjoint, the above inclusion means that

$$(\Delta^{1/4} t)^* (\Delta^{1/4} t)^- = |(\Delta^{1/4} t)^-|^2 = \Delta_{\phi\phi_0}^{1/2}.$$

Thus, $|(\Delta^{1/4} t)^-| = \Delta_{\phi\phi_0}^{1/4}$ is uniquely determined by ξ_ϕ .

Because of Lemma 2, $\Delta_{\phi_0}^{1/4}$ is essentially selfadjoint on $\mathfrak{M}'_0 \xi_0$. Thus the phase part v of the polar decomposition $(\Delta^{1/4} t)^- = v |(\Delta^{1/4} t)^-| = v \Delta_{\phi_0}^{1/4}$ is determined by

$$\begin{cases} v = 0 & \text{on } \mathfrak{R}(\Delta_{\phi_0}^{1/4})^\perp \quad (= (\mathfrak{M}'_0 \xi_0)^\perp), \\ v(\Delta_{\phi_0}^{1/4} x' \xi_0) = (\Delta^{1/4} t)^- x' \xi_0, & x' \in \mathfrak{M}'. \end{cases}$$

But, the right-hand side of the second equation is just $\Delta^{1/4} x' t \xi_0$ due to the fact that t is affiliated with the algebra. Lemma 5 thus implies that the phase part v is also uniquely determined by ξ_ϕ . (Q.E.D.)

Lemma 5 does not guarantee that t itself is uniquely determined by ξ_ϕ as an operator (unless $\mathfrak{M}$ is finite, [6, 8]). We will show this fact in the next section. We are thus convinced that Proposition 6 is a "correct" uniqueness property.

5. A counterexample. As stated in the last part of §4, we will show

PROPOSITION 7. *An operator t in the Theorem is not uniquely determined by a given ξ in $\mathfrak{P}^h$.*

In fact, by constructing a counterexample, we will show that t is not unique even when $\mathfrak{M}$ is a factor of type I_∞ . Our example is an improved version of the one given in [6].

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathfrak{M} = B(\mathcal{H})$ acts on the Hilbert space $\mathcal{K} = C_2(\mathcal{H})$ (of all Hilbert-Schmidt class operators on $\mathcal{H}$) as left multiplications. Then, $(\mathfrak{M}, \mathcal{K}, J = *, \mathfrak{P}^h = C_2^+(\mathcal{K}))$ is a standard form. In what follows, we will identify $C_2(\mathcal{H})$ with $\mathcal{H} \otimes \overline{\mathcal{H}}$ in the usual way. (For $\xi, \zeta \in \mathcal{H}$, $\xi \otimes \zeta$ is a rank one operator on $\mathcal{H}$ defined by $(\xi \otimes \zeta)\eta = (\eta | \zeta)_{\mathcal{H}} \xi$, $\eta \in \mathcal{H}$.) Thus, $\mathfrak{M}$ may be considered as the tensor product $B(\mathcal{H}) \otimes 1_{\overline{\mathcal{H}}}$.

We choose and fix a closed symmetric positive operator t on $\mathcal{K}$ satisfying $(t\xi | \xi)_{\mathcal{K}} \geq \|\xi\|_{\mathcal{K}}^2$, $\xi \in \mathfrak{D}(t)$, with two different positive selfadjoint extensions t_1 and t_2 . (In fact, suppose that b is a positive closed operator which is not essentially selfadjoint. Then, the Friedrich extension $(b + 1)_\infty$ and the Krein extension $(b + 1)_0$ of $b + 1$ are always different, [1].) We also fix a dense sequence $\{e_n\}_{n=1,2,\dots}$ in $\mathfrak{D}(t)$. Thanks to the well-known Schmidt orthogonalization process, we may and do assume that $\{e_n\}$ ($\subseteq \mathfrak{D}(t)$) is an orthonormal basis for $\mathcal{K}$. We then choose a sequence $\{\lambda_n\}_{n=1,2,\dots}$ of strictly positive numbers satisfying

$$(6) \quad \lambda_n \leq 1/n,$$

$$(7) \quad \lambda_n \|te_n\|_{\mathcal{K}} \leq 1/n^2,$$

$$(8) \quad \lambda_n \|te_n\|_{\mathcal{K}}^2 \leq 1/n^2.$$

At first, (6) means that $\xi_0 = \sum_{n=1}^\infty \lambda_n e_n \otimes e_n$ gives rise to an element in $\mathcal{K} = C_2(\mathcal{H}) = \mathcal{H} \otimes \overline{\mathcal{H}}$. This is a nonsingular positive Hilbert-Schmidt class operator on $\mathcal{K}$ so that $\xi_0 \in \mathfrak{P}^h$ is cyclic and separating. We now consider the closed positive operator $\tilde{t} = t \otimes 1$ affiliated with $\mathfrak{M} = B(\mathcal{H}) \otimes 1_{\overline{\mathcal{H}}}$ (see §VIII-10, [7], for tensor

products of unbounded operators). The vector ξ_0 belongs to $\mathfrak{D}(\tilde{t})$, and

$$\tilde{t}\xi_0 = \sum_{n=1}^{\infty} \lambda_n(te_n) \otimes e_n.$$

Indeed, the right-hand side is certainly a vector in $\mathcal{H}$ due to (7). Furthermore, (6) (resp. (7)) shows that $\sum_{n=1}^m \lambda_n e_n \otimes e_n$ (resp. $\tilde{t}(\sum_{n=1}^m \lambda_n e_n \otimes e_n) = \sum_{n=1}^m \lambda_n(te_n) \otimes e_n$) tends to ξ_0 (resp. $\sum_{n=1}^{\infty} \lambda_n(te_n) \otimes e_n$) as m goes to ∞ . Since $J(\xi \otimes \zeta) = (\xi \otimes \zeta)^* = \zeta \otimes \xi$, $\xi, \zeta \in \mathcal{H}$, we have

$$J\tilde{t}\xi_0 = \sum_{n=1}^{\infty} \lambda_n e_n \otimes (te_n).$$

Repeating the above arguments together with (7) and (8), we then have

$$J\tilde{t}\xi_0 \in \mathfrak{D}(\tilde{t}), \quad \tilde{t}J\tilde{t}\xi_0 = \sum_{n=1}^{\infty} \lambda_n(te_n) \otimes (te_n) \in \mathfrak{P}^q.$$

Thus, two different positive selfadjoint operators $\tilde{t}_1 = t_1 \otimes 1$ and $\tilde{t}_2 = t_2 \otimes 1$ affiliated with $\mathfrak{M}$ enjoy

$$\tilde{t}_1 J\tilde{t}_1 \xi_0 = \tilde{t}_2 J\tilde{t}_2 \xi_0 \quad \left(= \sum_{n=1}^{\infty} \lambda_n(te_n) \otimes (te_n) \in \mathfrak{P}^q \right).$$

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