LENGTH OF RAY-IMAGES UNDER CONFORMAL MAPS

V. KARUNAKARAN¹

ABSTRACT. Let $w = f(z)$ be regular and univalent in $|z| < 1$ with $f(0) = 0$. Suppose that f maps the unit disc onto a domain D . Let $l(r, \theta)$ be the length of the image curve of the ray joining $z = 0$ to $z = re^{i\theta}$ in D and $A(r) = \text{Sup}[|f(re^{i\theta})|^{-1}l(r, \theta)]$ where the supremum is taken over all starlike functions. In this paper we show that $A(r) \leq (1 + r)$.

Let $w = f(z)$ be regular and univalent in $|z| < 1$ and $f(0) = 0$. Suppose that f maps the unit disc $\{z: |z| < 1\}$ onto a domain D . Let $C(r, \theta)$ be the image in D of the ray joining $z = 0$ to $z = re^{i\theta}$ and let

$$l(r, \theta) = \int_0^r |f'(ue^{i\theta})| du$$

be its length. Gehring and Hayman [1] proved that if D is starlike with respect to $w = 0$

$$l(r, \theta) < k |f(re^{i\theta})|$$

where k is an absolute constant. Sheil-Small [4] showed that k can be taken as $(1 + \log 4)$ and if in addition f is starlike of order $1/2$ then k can be taken as $(1 + \log 2)$. His conjecture regarding the best possible constants in both these cases has been proved in the affirmative by Hall [2, 3]. In the later work Hall poses the problem of finding

$$A(r, \alpha) = \text{Sup} \left\{ |f(re^{i\theta})|^{-1} \int_0^r |f'(ue^{i\theta})| du \right\}$$

where the supremum is taken over all functions f which are starlike of order α and θ real. In this paper we show that $A(r, 0) \leq (1 + r)$ and it is likely that this is best possible up to rotations even though I am not able to establish this fact at present.

THEOREM. *Let f be starlike of order 0 in the unit disc then*

$$l(r, \theta) = \int_0^r |f'(ue^{i\theta})| du \leq (1 + r) |f(re^{i\theta})|.$$

PROOF. Without loss of generality let us suppose $\theta = 0$. Since f is starlike of order 0 we have

$$(1) \quad H(z) = \frac{zf'(zr)}{f(zr)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + zre^{-it}}{1 - zre^{-it}} dV(t)$$

Received by the editors December 30, 1980 and, in revised form, March 29, 1982.

1980 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C35.

Key words and phrases. Starlike, ray-image.

¹Research supported by a grant from Madurai University.

where $V(t)$ is increasing, $V(-\pi) + \pi = V(\pi) - \pi$. Let us put $h(z) = f(rz)$ so that $H(z) = zh'(z)/h(z)$. Now we can write

$$(2) \quad \frac{1}{u} \operatorname{Re} H(u) = \frac{\partial}{\partial u} \log |h(u)| = \int_0^\pi \frac{1 - u^2 r^2}{ur(1 - 2ur \cos t + u^2 r^2)} dW(t)$$

where $dW = dV(t) - dV(-t)$ is nonnegative and has total mass 1. We have to show that

$$(3) \quad \int_0^1 |h'(u)| du \leq (1 + r) |h(1)|.$$

We write

$$(4) \quad \begin{aligned} \int_0^1 |h'(u)| du &= \int_0^1 |H(u)| |h(u)| u^{-1} du \\ &= \int_0^1 \operatorname{Re} H(u) |h(u)| u^{-1} du + \int_0^1 (|H(u)| - \operatorname{Re} H(u)) |h(u)| u^{-1} du \end{aligned}$$

and observe that the first integral on the right side is $|h(1)|$ by (2). Further from (1) and (2) we have

$$(5) \quad \begin{aligned} &|H(u)| - \operatorname{Re} H(u) \\ &\leq \int_0^\pi \left\{ \frac{1 + ur}{\sqrt{(1 + u^2 r^2 - 2ur \cos t)}} - \frac{1 - u^2 r^2}{(1 + u^2 r^2 - 2ur \cos t)} \right\} dW(t) \end{aligned}$$

and (2) gives

$$\begin{aligned} \log |h(u)/h(1)| &= \int_1^u \frac{\partial}{\partial u} \log |h(u)| du \\ &= \int_0^\pi \log \frac{u(1 + r^2 - 2r \cos t)}{(1 + u^2 r^2 - 2ur \cos t)} dW(t) \end{aligned}$$

and so using Jensen's inequality [5, p. 24] we can get

$$(6) \quad \left| \frac{h(u)}{u} \right| \leq |h(1)| \int_0^\pi \frac{(1 + r^2 - 2r \cos t)}{(1 + u^2 r^2 - 2ur \cos t)} dW(t).$$

Hence

$$(7) \quad \begin{aligned} &\int_0^1 (|H(u)| - \operatorname{Re} H(u)) |h(u)| u^{-1} du \\ &\leq |h(1)| \int_0^\pi \int_0^\pi I(t, s) dW(s) dW(t) \\ &\leq \frac{1}{2} |h(1)| \int_0^\pi \int_0^\pi (I(t, s) + I(s, t)) dW(s) dW(t) \end{aligned}$$

where

$$(8) \quad \begin{aligned} I(t, s) &= \int_0^1 \left\{ \frac{1 + ur}{\sqrt{(1 + u^2 r^2 - 2ur \cos t)}} - \frac{1 - u^2 r^2}{(1 + u^2 r^2 - 2ur \cos t)} \right\} \\ &\quad \cdot \frac{(1 + r^2 - 2r \cos s)}{(1 + u^2 r^2 - 2ur \cos s)} du. \end{aligned}$$

In order to prove (3) it suffices to show

$$\int_0^\pi \int_0^\pi (I(t, s) + I(s, t)) dW(s) dW(t) < 2r.$$

Since dW is a probability measure, moreover I is even in both variables it will be sufficient for our result to show

$$I(s, t) + I(t, s) < 2r \quad \text{for } 0 < s < t < \pi$$

(see [3]). In the above integral in (8) we change ur as u and we get

$$(9) \quad I(t, s) = \frac{1}{r} \int_0^r \left\{ \frac{1+u}{\sqrt{(1+u^2-2u \cos t)}} - \frac{1-u^2}{(1+u^2-2u \cos t)} \right\} \cdot \frac{(1+r^2-2r \cos s)}{(1+u^2-2u \cos s)} du.$$

We observe that

$$(10) \quad I(t, s) = \frac{1+r^2-2r \cos s}{r} (J(t, s) - K(t, s))$$

where

$$(11) \quad J(t, s) = \int_0^r \frac{(1+u)}{\sqrt{(1+u^2-2u \cos t)}(1+u^2-2u \cos s)} du$$

and

$$(12) \quad K(t, s) = \int_0^r \frac{(1-u^2)}{(1+u^2-2u \cos t)(1+u^2-2u \cos s)} du.$$

We evaluate these integrals. In (11) we change the variable u to

$$y = (1-u)/\sqrt{(1+u^2-2u \cos t)}$$

and use $S = \sin(s/2)$ and $T = \sin(t/2)$ to get

$$(13) \quad J(t, s) = \frac{1}{2S\sqrt{(T^2-S^2)}} \left[\arctan(T^2-S^2/S) - \arctan(\sqrt{(T^2-S^2)}(1-r)/S\sqrt{(1+r^2-2r \cos t)}) \right].$$

Now for simplification purposes we write

$$(14) \quad k = \sqrt{(T^2-S^2)}/T,$$

$$1-x^2 = (1-r)^2/(1-r)^2 + 4rT^2, \quad 0 < x \leq 2\sqrt{r}/1+r < 1.$$

The relations (14) imply that

$$(15) \quad ((1-r)^2/(1-r)^2 + 4rS^2) = (1-x^2)/(1-k^2x^2),$$

$$T^2 = (1-r)^2x^2/4r(1-x^2), \quad S^2 = (1-r)^2(1-k^2)x^2/4r(1-x^2).$$

The relations (14) and (15) show that

(16)

$$\begin{aligned}
 & \frac{1 + r^2 - 2r \cos s}{r} J(t, s) \\
 &= \frac{2(1 - k^2 x^2)}{k x^2 \sqrt{1 - k^2}} \left(\arctan(k/\sqrt{1 - k^2}) \right. \\
 & \quad \left. - \arctan k\sqrt{(1 - x^2)/\sqrt{1 - k^2}} \right) \sqrt{1 - k^2} \\
 &= \frac{2(1 - k^2 x^2)}{x^2(1 - k^2)} \int_0^x \frac{(1 - k^2)t}{(1 - k^2 t^2)\sqrt{1 - t^2}} dt \\
 &= \frac{2(1 - k^2 x^2)}{x^2(1 - k^2)} \left\{ \int_0^x \frac{t}{\sqrt{1 - t^2}} dt - \sum_{n=1}^{\infty} k^{2n} \int_0^x t^{2n-1} \sqrt{1 - t^2} dt \right\}; \\
 & K(t, s) = \frac{1}{4(T^2 - S^2)} \log \frac{(1 - r)^2 + 4rT^2}{(1 - r)^2 + 4rS^2}
 \end{aligned}$$

and

$$(17) \quad \frac{1 + r^2 - 2r \cos s}{r} K(t, s) = -\frac{(1 - k^2 x^2)}{k^2 x^2} \log(1 - k^2 x^2).$$

Interchanging the roles of s and t we can also get

$$\begin{aligned}
 J(s, t) &= \frac{1}{2T\sqrt{T^2 - S^2}} \\
 & \cdot \log \left\{ \frac{T + \sqrt{(T^2 - S^2)}}{S} \frac{T - \sqrt{(T^2 - S^2)}\sqrt{(1 - x^2)}/\sqrt{(1 - k^2 x^2)}}{\sqrt{(T^2 - (T^2 - S^2)(1 - x^2)/1 - k^2 x^2)}} \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 (18) \quad \frac{1 + r^2 - 2r \cos t}{r} J(s, t) &= \frac{2}{kx^2} \log \frac{1 + k}{\sqrt{(1 - k^2 x^2)} + k\sqrt{(1 - x^2)}} \\
 &= \frac{2}{x^2} \int_0^x \frac{t}{\sqrt{(1 - t^2)}\sqrt{(1 - k^2 t^2)}} dt \\
 &= \frac{2}{x^2} \sum_{n=0}^{\infty} C_n k^{2n} \int_0^x t^{2n+1} (1 - t^2)^{-1/2} dt
 \end{aligned}$$

where

$$(19) \quad C_n = \Gamma(n + 1/2)/\Gamma(1/2)\Gamma(n + 1).$$

Now $K(s, t) = K(t, s)$ and so

$$(20) \quad \frac{1 + r^2 - 2r \cos t}{r} K(s, t) = -\frac{1}{k^2 x^2} \log(1 - k^2 x^2).$$

Thus using (16), (17), (18) and (20) we get
(21)

$$I(t, s) + I(s, t) = \frac{2(1 - k^2 x^2)}{(1 - k^2)x^2} \left\{ \int_0^x \frac{t}{\sqrt{1 - t^2}} dt - \sum_{n=1}^{\infty} k^{2n} \int_0^x t^{2n-1} \sqrt{1 - t^2} dt \right\} \\ + \frac{2}{x^2} \sum_{n=0}^{\infty} C_n k^{2n} \int_0^x \frac{t^{2n+1}}{\sqrt{1 - t^2}} dt + \left(\frac{2 - k^2 x^2}{k^2 x^2} \right) \log(1 - k^2 x^2).$$

If we write $I(t, s) + I(s, t)$ as a power series in k of the form $\sum_{n=0}^{\infty} \Phi_n(x) k^{2n}$ we find after some work that

$$(22) \quad \Phi_n(x) = -2 \int_0^x \frac{t^{2n-1}}{\sqrt{1 - t^2}} dt + \frac{2 + 2C_n}{x^2} \int_0^x \frac{t^{2n+1}}{\sqrt{1 - t^2}} dt - \frac{2x^{2n}}{n+1} + \frac{x^{2n}}{n}.$$

Expanding $\Phi_n(x)$ also in a power series we find that

$$(23) \quad \Phi_n(x) = \sum_{j=0}^{\infty} C_j \left[\frac{1 + C_n}{j + n + 1} - \frac{1}{j + n} \right] x^{2j+2n} - \frac{2x^{2n}}{n+1} + \frac{x^{2n}}{n}.$$

From (23) it is clear that $\Phi_n(0) = 0$ and that the coefficient of x^{2j+2n} ($j \geq 1$) is $((n+j)C_n - 1)C_j$. Now

$$((n+j)C_n - 1)C_j \geq ((n+1)C_n - 1)C_j$$

and $(n+1)C_n$ can be shown by induction to be greater than or equal to 1 using (19). Hence the coefficients are positive and in particular $\Phi_n(x)$ has at most one stationary value which in this case must be a minimum. Hence

$$\Phi_n(x) \leq \text{Max}(\Phi_n(0), \Phi_n(x_0))$$

where x_0 is the maximum possible value for x namely $x_0 = 2\sqrt{r}/(1+r)$. However

$$\Phi_n(x_0) \leq \text{Max}(\Phi_n(0), \Phi_n(1))$$

since x_0 may be less or greater than the value of x at which $\Phi_n(x)$ is minimum. But from (22)

$$\Phi_n(1) = -\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} + \frac{n\Gamma(n)\Gamma(1/2)}{\Gamma(n+(1/2))(n+(1/2))} + \frac{2}{2n+1} - \frac{2}{n+1} + \frac{1}{n} \\ = \frac{1}{(2n+1)} \left\{ 2 - \frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+(1/2))} \right\} - \frac{2}{n+1} + \frac{1}{n}.$$

Since $(n+1/4)\Gamma^2(n+1/2)/\Gamma^2(n+1)$ increases to 1 we have

$$(24) \quad \Gamma(n+1/2)/\Gamma(n+1) \leq 2/\sqrt{1+4n}.$$

Using (24) and the fact that $\pi > 2$ one can easily show that $\Phi_n(1) < 0$ for $n \geq 5$. For $n < 5$ one can check this inequality directly. Hence $\Phi_n(x) < 0$. For all $n \geq 1$. Thus we get

$$(25) \quad I(s, t) + I(t, s) \leq \Phi_0(x) = \frac{4}{x^2} - 4 \frac{\sqrt{1-x^2}}{x^2} - 2.$$

Now

$$\frac{d}{dx}(\Phi_0(x)) = \frac{-8}{x^3} + \frac{4(2-x^2)}{x^3\sqrt{1-x^2}} > 0.$$

Hence $\Phi_0(x)$ increases and has a maximum at $x = x_0$. Thus

$$(26) \quad \Phi_0(x) \leq \Phi_0(x_0) = 2r.$$

We use (26) in (25) and conclude from (7) that

$$(27) \quad \int_0^1 [|H(u)| - \operatorname{Re} H(u)] \left| \frac{h(u)}{u} \right| du \leq r |h(1)|$$

and as already observed

$$(28) \quad \int_0^1 \operatorname{Re} H(u) |h(u)| u^{-1} du = |h(1)|.$$

Now (27) and (28) and (4) gives (3). Hence the result.

REFERENCES

1. F. W. Gehring and W. K. Hayman, *An inequality in the theory of conformal mapping*, J. Math. Pures Appl. **41** (1963), 353–361.
2. R. R. Hall, *The length of ray images under starlike mappings*, Mathematika **23** (1976), 147–150.
3. ———, *A conformal mapping inequality for starlike functions of order 1/2*, Bull. London Math. Soc. **12** (1980), 119–126.
4. T. Sheil-Small, *Some conformal mapping inequalities for starlike and convex functions*, J. London Math. Soc. (2) **1** (1969), 577–587.
5. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London and New York, 1968.

SCHOOL OF MATHEMATICS, MADURAI UNIVERSITY, MADURAI-625021, INDIA