THE CONJUGACY PROBLEM FOR GRAPH PRODUCTS WITH CYCLIC EDGE GROUPS

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ABSTRACT. A graph product is the fundamental group of a graph of groups. Amongst the simplest examples are HNN groups and free products with amalgamation.

The conjugacy problem is solvable for recursively presented graph products with cyclic edge groups over finite graphs if the vertex groups have solvable conjugacy problem and the sets of cyclic generators in them are semicritical. For graph products over infinite graphs these conditions are insufficient: a further condition ensures that graph products over infinite graphs of bounded path length have solvable conjugacy problem. These results generalise the known ones for HNN groups and free products with amalgamation.

1. Introduction. Groups which are graph products (fundamental groups of graphs of groups in the terminology of Bass and Serre) have attracted considerable attention, in view of their utility in combinatorial group theory. In particular, decision problems for HNN extensions and free products with amalgamation—the simplest graph products—have been of interest. The conjugacy problem for graph products is generally unsolvable (Miller [10]), but Lipschutz [9] gives conditions ensuring solvability of the conjugacy problem for free products with cyclic amalgamated subgroups and Hurwitz [7] for HNN extensions with cyclic associated subgroups.

This paper considers the conjugacy problem for graph products with cyclic edge groups, as part of the more general conjugacy problem for (Brandt) groupoids. Note that the defining graph here is slightly different from that considered in the Bass-Serre theory, so that an edge group there is regarded as a group at a source vertex in the terms of this paper.

The semicriticality conditions of Lipschutz and Hurwitz for cyclic generators are amended and extended below. It is then shown that under the anticipated generalisation of their conditions, the conjugacy problem is always solvable only for graph products over finite graphs. This restriction is due to the fact that additional graph-theoretic decision problems arise naturally in this context. For graph products over infinite graphs, a further condition is given which ensures that the conjugacy problem is solvable for graph products over graphs with either bounded path length or else finitely many sources and an infinite cyclic group at each.

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Terminology below follows that of [6]. The reader is referred to [2] for the theory of computability, to [5] for the basic constructions of the theory of groupoids, to [11] for the graph-theoretical background and to [12] for the Bass-Serre theory.

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2. The conjugacy problem for groupoids. For a presentation of a (countable) groupoid Γ , the conjugacy problem is that of determining if there is an algorithm to decide for an arbitrary pair of elements of Γ whether or not they are conjugate in Γ . Since conjugate elements of Γ must be loops (closed edge-sequences in the underlying graph of Γ) and must lie in the same connected component of (the underlying graph of) Γ , the conjugacy problem splits into three subproblems.

Question 2.1. Is there an algorithm to determine whether or not an arbitrary element of Γ is a loop?

Question 2.2. Is there an algorithm to determine whether or not an arbitrary pair of elements lie in the same component of Γ ?

Question 2.3. Is there an algorithm to determine whether or not an arbitrary pair of loops in the same component of Γ are conjugate in Γ ?

The third question translates the conjugacy problem for groups into groupoid terms, but the first two seek information about the general structure of Γ as a groupoid. In [6, §3] it is shown that Question 2.1 has a sensible interpretation and then is answerable in the affirmative if Γ is recursively presented.

DEFINITION 2.4. A groupoid Γ is *recursively presented* if it has a presentation $(\Gamma = \langle X : R \rangle; \text{Id } \Gamma; \lambda, \rho: X \rightarrow \text{Id } \Gamma)$ for which

- (i) the set of *generators* X is recursively enumerable (r.e.);
- (ii) the set of *relators* R is r.e.;
- (iii) the set of *identities* $\text{Id } \Gamma$ is recursive;
- (iv) the *left* and *right identity maps* λ, ρ are partially recursive. $\square$

If Γ is recursively presented then Question 2.2 may be rephrased as the graph component problem for its underlying graph $U(\Gamma)$, the image of Γ under the forgetful functor from groupoids to directed graphs.

Question 2.5 (The component problem for (countable) graph γ). Is there an algorithm to determine whether or not an arbitrary pair of vertices of γ lie in the same connected component of γ ?

For this problem to make sense, γ must be suitably presented. This prompts the following definition (cf. [6, 3.4(a)]).

DEFINITION 2.6. A graph γ is *recursive* if

- (i) the set of *vertices* $V = V(\gamma)$ is recursive;
- (ii) the set of *edges* $E = E(\gamma)$ is recursive;
- (iii) the *incidence map* $\delta: E \rightarrow V \times V$ (with domain E) is partially recursive. If γ is directed, $\delta = \lambda \times \rho$ where $\lambda, \rho: E \rightarrow V$ are the *source* and *sink vertex maps*, respectively. $\square$

Obviously, if γ is (known to be) connected, its component problem is solvable whether it is recursive or not. Further, it is clear that for recursive graphs with finite (known) number of edges, the component problem is also solvable, for if the *radius* d

neighbourhood of vertex v is defined as

$$N_d(v) = \{w \in V(\gamma) : \text{there is an arc of length } \leq d \text{ joining } w \text{ and } v\},$$

then there is an algorithm to find $N_d(v)$ for each $d \geq 1$, and when $N_d(v) = N_{d+1}(v)$, the finite set of all vertices in the same component as v has been found.

However, for recursive graphs with infinite edge set, any process involving checking infinitely many edges sequentially will not terminate. Whilst in practice it is likely that enough information is available about γ for its component problem to be solvable, recursive graphs with unsolvable component problem certainly exist. For example, define the *halting problem graph* γ_M as follows. Let M be a Turing machine for which the halting problem is unsolvable (cf. [2, C. 2, §1]). The vertices of γ_M are the configurations of M and the edges are the ordered pairs (v, v') for which v' is the unique successor configuration of v . Since M is based on a finite alphabet, γ_M is recursive. By [2, C. 2.1.2] there is no algorithm to determine whether or not a given configuration v of $V(\gamma_M)$ lies in the same component as a halting configuration h .

Hence in the absence of an oracle, the answer to Question 2.5 for possibly disconnected graphs in which arbitrarily long paths may exist, must be “no”. Halin [4] has characterised an intersecting class of infinite graphs: those without an infinite path.

If the component diameters (maximum arc lengths) of γ are known to be bounded, the component problem may still be solvable.

Question 2.7 (The radius d neighbourhood problem for γ). Given $d \geq 1$, is there an algorithm to determine for an arbitrary pair v, w of vertices of γ , whether or not $w \in N_d(v)$?

LEMMA 2.8. *If γ is a recursive graph with $E(\gamma)$ infinite, and its components are known to have finite diameters bounded by $d \geq 1$, then the component problem for γ is solvable if and only if the radius d neighbourhood problem is solvable for γ . $\square$*

Consider now the case that a groupoid Γ is (presented as) a mapping cylinder $G = m(D, A)$ of a group diagram (D, A) with monic edge homomorphisms, over a connected directed graph $D = (E, V)$. That is, (D, A) consists of a group A_v with identity 1_v for each v in V and a group monomorphism $A_e: A_{\lambda e} \rightarrow A_{\rho e}$ for each e in E . If A_v has a presentation $\langle X_v: R_v \rangle$ for each v in V , G has a presentation $(G = \langle X: R \rangle; \text{Id } G; \lambda, \rho: X \rightarrow \text{Id } G)$ with

$$X = \left(\bigcup X_v \right) \cup E^*, \quad \text{where } E^* = \{t_e: e \in E\},$$

$$R = \left(\bigcup R_v \right) \cup \{t_e^{-1}xt_e = A_e(x), e \in E, x \in X_{\lambda e}\},$$

$$\text{Id } G = \bigcup \{1_v\},$$

$$\lambda x = \rho x = 1_v \quad \text{for } x \in X_v, \quad \lambda t_e = 1_{\lambda e} \quad \text{and} \quad \rho t_e = 1_{\rho e}.$$

In [6, 3.5] it is shown that G is recursively presented and E^* is recursive if and only if (D, A) is recursively presented [6, 3.4(b)].

DEFINITION 2.9. The group diagram (D, A) is *recursively presented* if

- (i) D is recursive;
- (ii) A_v is recursively presented for each v in V and uniformly given from V ;
- (iii) A_e is partially recursive for each e in E and uniformly given from E . $\square$

Hereafter it will be assumed that G is (presented as) the mapping cylinder groupoid of a recursively presented group diagram (D, A) with monic edge homomorphisms, such that each source group A_v (i.e. $v = \lambda e$ for at least one e in E) is presented as a cyclic group on a single generator k_v .

The graph product of G (the loop group at any selected vertex) will be denoted G_0 .

Such a mapping cylinder G is recursively presented and $U(G)$ is connected since D is, so the answers to Questions 2.1 and 2.2 are both "yes". The next section looks at Question 2.3 for G .

3. The conjugacy problem for mapping cylinder groupoids. In [9], Lipschutz proves the conjugacy problem is solvable for any free product with cyclic amalgamated subgroups for which the factors have solvable conjugacy problem and the generators of the amalgamated subgroups are what he terms "semicritical". In [7], Hurwitz proves the conjugacy problem is solvable for any HNN extension with cyclic associated subgroups for which the base group has solvable conjugacy problem and the generators of the associated groups are "mutually semicritical".

Hurwitz' condition of mutual semicriticality is amended and generalised below. The conjugacy relation in a group or groupoid will be symbolised $\sim$; where necessary, a subscript indicating context will be added.

DEFINITION 3.1. Let H be a subset of group A . Then H is *semicritical* if the following conditions all hold:

- (i) for each h in H , $h^n \sim h^m$ implies $h^n = h^m$, and for $k \neq h$ in H , $h^n \sim k^m$ implies $h^n = k^m = 1$;
- (ii) for each a in A it can be decided whether or not a is conjugate to a power of an element in H ;
- (iii) for each (a, b) in $A \times A$ and (h, k) in $H \times H$ it can be decided whether or not there exist powers h^n, k^m not both 1 such that $h^n a k^m = b$. $\square$

For example, if $\mathbf{Z}_2$ is the cyclic group of order 2, any subset of $\mathbf{Z}_2^n$ is semicritical. By (3.1(i)) any semicritical subset not containing 1 of an infinite cyclic group must be a singleton. If A has solvable conjugacy problem and H is recursive then for each affirmative answer to (3.1(ii)) there is an algorithm to determine suitable powers of elements of H .

DEFINITION 3.2. Let $G = m(D, A)$. For each v in V define $H_v \subseteq A_v$ by

$$H_v = \{k_{\lambda e} : \lambda e = v, e \in E\} \cup \{A_e(k_{\lambda e}) : \rho e = v, e \in E\}. \quad \square$$

Sufficient conditions can now be given to ensure the conjugacy problem is solvable for pairs of loops in G , provided at least one of them has nonzero reduced length [6, 2.1].

THEOREM 3.3. Suppose G satisfies the following conditions.

- (i) A_v has uniformly solvable conjugacy problem for every v in V .
- (ii) H_v is a uniformly semicritical recursive set for every v in V .

Then the conjugacy problem is solvable for any two elements of which at least one has nonzero reduced length.

PROOF. Since under these conditions $\langle A_e(k_{\lambda e}) \rangle$ has uniformly solvable extended word problem in $A_{\rho e}$ for every e in E , the conditions of [6, 3.6] hold for G , so the

word problem for G is solvable, and the process of finding a cyclically reduced loop conjugate to a given loop of G is algorithmic. By the Conjugacy Theorem [6, 2.6] it is necessary only to solve the conjugacy problem for distinct pairs of nontrivial cyclically reduced loops g and h , where $g = a_1 s_1 \cdots a_n s_n$, $h = b_1 s_1 \cdots b_n s_n$, a_i and b_i lie in the same vertex group, s_i is an edge-symbol or its inverse, and $n \geq 1$. Determine algorithmically the minimum positive integer k such that $n = qk$ and the sequence $s_1, s_2, \dots, s_n$ is the sequence $s_1, s_2, \dots, s_k$ repeated q times. Then $g \sim_G h$ if and only if there exist j , $1 \leq j \leq q$, and an integral solution $(p_1, \dots, p_n)$ to the following system of equations in G , where $l = jk$:

$$\begin{aligned} b_{l+1} x_1^{p_1} &= y_n^{p_n} a_1 \\ b_{l+2} x_2^{p_2} &= y_1^{p_1} a_2 \\ &\vdots \\ b_n x_{n-l}^{p_{n-l}} &= y_{n-l-1}^{p_{n-l-1}} a_{n-l} \\ b_1 x_{n-l+1}^{p_{n-l+1}} &= y_{n-l}^{p_{n-l}} a_{n-l+1} \\ &\vdots \\ b_{l-1} x_{n-1}^{p_{n-1}} &= y_{n-2}^{p_{n-2}} a_{n-1} \\ b_l x_n^{p_n} &= y_{n-1}^{p_{n-1}} a_n, \end{aligned}$$

where

$$x_i = \begin{Bmatrix} k_{\lambda_e} \\ A_e(k_{\lambda_e}) \end{Bmatrix} \quad \text{and} \quad y_i = \begin{Bmatrix} A_e(k_{\lambda_e}) \\ k_{\lambda_e} \end{Bmatrix} \quad \text{if } s_i = \begin{Bmatrix} t_e \\ t_e^{-1} \end{Bmatrix}, \quad \text{for } 1 \leq i \leq n.$$

This corresponds to the system (*) of equations of Hurwitz [7, p. 2], and the obvious modifications of the remainder of his proof suffice. $\square$

For cyclically reduced loops in the vertex groups A_v , the conjugacy problem involves questions about the connectivity of elements in $\cup H_v$, and prompts the next definition.

DEFINITION 3.4. The *unfolded graph* $\mathcal{U}(G)$ of G is the directed graph with edge set $\mathcal{E} = E = \{e \in D\}$, vertex set $\mathcal{V} = \cup H_v$ and incidence map $\delta(e) = (k_{\lambda_e}, A_e(k_{\lambda_e})) \in H_{\lambda_e} \times H_{\rho_e}$, for all $e \in \mathcal{E}$. $\square$

The unfolded graph $\mathcal{U}(G)$ is a possibly disconnected subgraph of $U(G)$, recursive if and only if $\mathcal{V}$ is recursive (for example under conditions (3.3)). Its properties depend on the defining graph D of G and the group monomorphisms $\{A_e: e \in E\}$. For example, if D_1 has $V(D_1) = \{0, 1, 2\}$ and $E(D_1) = \{e = (1, 0), f = (2, 0)\}$ then $\mathcal{U}(G)$ is either the connected graph $k_1 \xrightarrow{e} A_e(k_1) \xrightarrow{f} k_2$ or the disconnected graph $k_1 \xrightarrow{e} A_e(k_1), k_2 \xrightarrow{f} A_f(k_2)$ depending on whether or not $A_e(k_1) = A_f(k_2)$. If E is infinite, then $\mathcal{V}$ may be infinite even if V is finite.

LEMMA 3.5. Under the conditions (3.3) the conjugacy problem is solvable for any two elements in G of zero reduced length if and only if the component problem for $\mathcal{U}(G)$ is solvable.

PROOF. Suppose g and h are distinct cyclically reduced loops of G with $\|g\| = \|h\| = 0$, so that $g \in A_v$ and $h \in A_w$ for (known) v and w ; if $v = w$ assume $g \sim_v h$.

Then $g \sim_G h$ if and only if there exist x_v in H_v , x_w in H_w and integer p (all uniquely determined) and a product $u^* = s_1 s_2 \cdots s_n$ in G of edge-symbols and their inverses alone, such that $n \geq 1$, $g \sim_v x_v^p$, $h \sim_w x_w^p$ and $u^* x_v = x_w u^*$. The result follows from the one-to-one correspondence between such words u^* and edge-sequences in $\mathcal{Q}(G)$. $\square$

If D is infinite, $\mathcal{Q}(G)$ is infinite and its component problem may be unsolvable without further information about (D, A) . On the other hand, if D has edge set of known finite size, so has $\mathcal{Q}(G)$ and the expected generalisation of Lipschutz and Hurwitz' work to finite graph products with cyclic sources follows.

THEOREM 3.6. *Let G_0 be (presented as) the loop group of a mapping cylinder G over a finite graph D with an edge set of known size. If*

- (i) A_v has solvable conjugacy problem for every v in V , and
- (ii) H_v is semicritical for every v in V ,

then G , and hence G_0 , has solvable conjugacy problem. $\square$

If D is infinite, it is natural to search for properties of D which ensure that $\mathcal{Q}(G)$ is well-behaved for all G over D satisfying conditions (3.3). For such a G , every path in $\mathcal{Q}(G)$ corresponds to a path in D , though not always vice-versa. Any property which holds for $\mathcal{Q}(G)$ for all G over D satisfying (3.3(i), (ii)), must hold for D , since in particular it must hold for G_D , which has the integers $\mathbf{Z}$ at each vertex and the identity automorphism at every edge, and $\mathcal{Q}(G_D) \cong D$ as graphs. Thus a necessary condition for the component diameters of $\mathcal{Q}(G)$ to be bounded for every G over D satisfying (3.3(i), (ii)) is that D have finite diameter. This condition is insufficient; a counterexample is $G = m(D_2, A)$ where $V(D_2) = \{i, (i, j) : i \geq 0, i \geq j \geq 1\}$, $E(D_2) = \{e_i = (0, i), i \geq 1; f_{ji} = ((i, j), i), g_{ji} = ((i, j), i), f_{ji} \neq g_{ji}, i \geq j \geq 1\}$, $A_i = \mathbf{Z}_2^{i+2}$ for $i \geq 1$, $A_0 = A_{(i,j)} = \mathbf{Z}_2$, and A_{e_i} , $A_{f_{ji}}$ and $A_{g_{ji}}$ are the injection homomorphisms into the 1st, $(j+1)$ st and $(j+2)$ nd components of A_i , respectively. In this case D_2 has diameter 4 but contains arbitrarily long paths. However, if D has bounded path length, so must $\mathcal{Q}(G)$, and (2.8) may be invoked.

THEOREM 3.7. *Let G_0 be (presented as) the loop group of a mapping cylinder G over an infinite graph D with maximum path length d (finite). Under the conditions (3.3) the conjugacy problem for G (and hence G_0) is solvable if and only if the radius d neighbourhood problem for $\mathcal{Q}(G)$ is solvable.* $\square$

This is also true if D has only finitely many sources (though in this case D may have arbitrarily long paths) provided each source group is infinite cyclic.

THEOREM 3.8. *Let G_0 be (presented as) the loop group of a mapping cylinder G over an infinite graph D with s sources in which each source group is infinite cyclic. Under the conditions (3.3) the conjugacy problem for G (and hence G_0) is solvable if and only if the radius $2s$ neighbourhood problem for $\mathcal{Q}(G)$ is solvable.*

PROOF. Let P be an arc in $\mathcal{Q}(G)$. No nonsource vertices in P are adjacent in P , so the length of P is less than or equal to twice the number of source vertices in P . Hence the diameter of any component of $\mathcal{Q}(G)$ is $\leq 2s$. $\square$

For graph products to which none of (3.6), (3.7) or (3.8) apply, the answer to (2.3) must be "no", unless additional information allows the conjugacy problem to be investigated on a case-by-case basis. Finally, solutions obtained above are not presentation-free: a group may have two presentations as graph products of which only one has solvable conjugacy problem; the other need not have even solvable word problem.

Semicriticality is a major restriction. Lipschutz [9] shows that nontrivial singleton subsets of finitely generated free groups are semicritical, and Comerford and Truffault [3] the same for elements of odd or infinite order in sixth groups. Larsen [8] shows that (3.1.i) is decidable for many cyclic HNN groups; elsewhere, semicriticality must be proved as required.

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