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ABSTRACT. Let L be a finite dimensional Lie algebra over a field k of characteristic zero, $D(L)$ the quotient division ring of $U(L)$. We compare the center $Z(D(L))$ with $Z(D(H))$ where H is an ideal of L of codimension one.

Let k be a field of characteristic zero. We first recall the following result due to Bernat [1; 3, p. 134–135].

LEMMA 1. *Let A be an associative algebra over k with a unit element, B a subalgebra containing 1, $x \in A$ an element symmetric (i.e. $[x, B] \subset B$) and transcendental with respect to B such that $A = B[x]$. Hence every element $a \in A$ can be considered as a polynomial in x with coefficients in B . We denote by $\deg a$ the degree (in x) of a . We assume further that A and B have quotient division rings K and L (so that L is the division subring of K generated by B) and that Δ is a set of derivations of A leaving stable B such that $\deg(Ex) \leq 1$ for all $E \in \Delta$. We extend the elements of Δ to derivations of K and denote by K^Δ the set of elements of K which are annihilated by all $E \in \Delta$. Then the following hold:*

- (1) x is symmetric and transcendental with respect to L and if $a \in A$ then $\deg_B a = \deg_L a$. $L \cap A = B$ and the algebra $L[x]$ has a quotient division ring which is K .
- (2) K^Δ is the quotient division ring of $L[x]^\Delta$.
- (3) Suppose $K^\Delta \neq L^\Delta$ (i.e. $L[x]^\Delta \not\subset L$) and let z be an element of $L[x]^\Delta$ of minimal degree > 0 . Then z is transcendental over L^Δ , $L[x]^\Delta = L^\Delta[z]$ and $K^\Delta = L^\Delta(z)$.

COROLLARY. *Let us assume in addition to the hypotheses of the lemma that $\deg(Ex) = 0$ for all $E \in \Delta$. Then z is of the form $cx + d$ with $c \in L^\Delta$, $c \neq 0$, and $d \in L$.*

PROOF. Choose $u \in L[x]^\Delta \setminus L$. We can find $c_i \in L$ such that $u = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ with $c_n \neq 0$ and $n \geq 1$. Let $E \in \Delta$; then

$$Eu = (Ec_n)x^n + c_n Ex^n + (Ec_{n-1})x^{n-1} + c_{n-1} Ex^{n-1} + \cdots + Ec_0.$$

On the other hand, for each $m \geq 1$ we have that

$$Ex^m = \sum_{i=1}^m x^{m-i} (Ex) x^{i-1} = m(Ex)x^{m-1} + \text{terms of lower degree}.$$

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Hence,

$$(*) \quad Eu = (Ec_n)x^n + (nc_nEx + Ec_{n-1})x^{n-1} + \text{terms of lower degree}.$$

This is a polynomial with coefficients in L since $Ec_i \in L$ and $Ex \in B \subset L$. Now, $u \in L[x]^\Delta$ implies that $Eu = 0$ for all $E \in \Delta$. Taking into account that x is transcendental with respect to L , we obtain $Ec_n = 0$ and $nc_nEx + Ec_{n-1} = 0$ so that $E(nc_nx + c_{n-1}) = 0$ for all $E \in \Delta$. Consequently, $nc_nx + c_{n-1} \in L[x]^\Delta$ with $c_n \in L^\Delta$, $c_n \neq 0$. Therefore $\deg z = 1$. Finally, it follows easily that $z = cx + d$ for some $c \in L^\Delta$, $c \neq 0$, $d \in L$.

REMARK. If we assume furthermore that $A^\Delta \neq B^\Delta$ (i.e. $A^\Delta \not\subset B$ and a fortiori $K^\Delta \neq L^\Delta$), then a similar argument shows that we can find an element $z = b_1x + b_2 \in A^\Delta$ with $b_1 \in B^\Delta$, $b_1 \neq 0$, and $b_2 \in B$ such that $L[x]^\Delta = L^\Delta[z]$ and $K^\Delta = L^\Delta(z)$.

Next, let L be a finite dimensional Lie algebra over k , $U(L)$ its universal enveloping algebra and $D(L)$ the quotient division ring of $U(L)$. We denote by $Z(U(L))$ (resp. $Z(D(L))$) the center of $U(L)$ (resp. $D(L)$). We now recall the following result due to Dixmier [2, p. 339].

LEMMA 2. Let H be an ideal of L of codimension one. If $Z(U(L)) \not\subset U(H)$ then $Z(U(H)) \subset Z(U(L))$ and $Z(U(H)) = Z(U(L)) \cap U(H)$.

It follows that we have either $Z(U(L)) \subset Z(U(H))$ or $Z(U(H)) \subset Z(U(L))$.

We are now in a position to prove the following

THEOREM. Let H be an ideal of L of codimension one, $x \in L \setminus H$. Then we have:

1. (a) If $Z(D(L)) \subset Z(D(H))$ then $Z(U(L)) \subset Z(U(H))$.

(b) If $Z(D(H)) \subset Z(D(L))$ then $Z(U(H)) \subset Z(U(L))$.

2. (a) If $Z(U(L)) \subset Z(U(H))$ then $Z(D(L)) \subset Z(D(H))$.

(b) If $Z(U(H)) \subset Z(U(L))$ then $Z(D(H)) \subset Z(D(L))$ and in this case $Z(D(L))$ is a purely transcendental extension of $Z(D(H))$ of degree one. Indeed, there exists an element $z = b_1x + b_2 \in Z(U(L))$, transcendental over $Z(D(H))$, with $b_1 \in Z(U(H))$, $b_1 \neq 0$, and $b_2 \in U(H)$ and such that $Z(D(L))$ is generated as a field by z and $Z(D(H))$. Finally, $Z(D(H)) = Z(D(L)) \cap D(H)$.

PROOF. 1. (a) $Z(U(L)) \subset Z(D(L)) \subset Z(D(H)) \subset D(H)$. Hence $Z(U(L)) \subset D(H) \cap U(L) = U(H)$. Consequently $Z(U(L)) \subset Z(U(H))$.

(b) $Z(U(H)) \subset Z(D(H)) \subset Z(D(L))$. Therefore $Z(U(H)) \subset Z(D(L)) \cap U(L) = Z(U(L))$.

2. Put $A = U(L)$, $B = U(H)$ and $\Delta = \text{ad } L$. Then A, B, x and Δ satisfy the conditions of Lemma 1. Indeed, $U(L)$ and $U(H)$ have quotient division rings $D(L)$ and $D(H)$. Since H is an ideal of L , every derivation $E \in \text{ad } L$ leaves stable $U(H)$. This holds in particular for $\text{ad } x$, which means that x is symmetric with respect to $U(H)$. By the Poincaré-Birkhoff-Witt Theorem x is also transcendental over $U(H)$ and $U(L) = U(H)[x]$. Hence each element of $U(L)$ can be written as a polynomial in x with coefficients in $U(H)$ (say on the left). Since H is an ideal of L of codimension one, we have that $[L, L] \subset H$ and so $\deg(Ex) = 0$ for all $E \in \Delta = \text{ad } L$. We now proceed with the proof.

(a) Suppose $Z(U(L)) \subsetneq Z(U(H))$. Take $y \in Z(U(H)) \setminus Z(U(L))$ and consider the derivation $E = \text{ad } y$ in $U(L)$. Clearly, $E(U(H)) = 0$ and $Ex = -\text{ad } x(y) \in U(H)$, $Ex \neq 0$. Take now $w \in Z(D(L))$. We can choose elements $u, v \in U(L)$, $v \neq 0$, with $w = uv^{-1}$ and such that $\deg v$ is minimal. From $u = wv$ we obtain $Eu = (Ew)v + wEv = wEv$. Write $v = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ with $c_i \in U(H)$ and $c_n \neq 0$. Now suppose $Ev \neq 0$. This forces $n \geq 1$. Then formula (*) gives us

$$\begin{aligned} Ev &= (Ec_n)x^n + (nc_n Ex + Ec_{n-1})x^{n-1} + \text{terms of lower degree} \\ &= nc_n(Ex)x^{n-1} + \text{terms of lower degree,} \end{aligned}$$

since $Ec_i = 0$ for all i . In particular $\deg(Ev) = \deg v - 1$, which contradicts the minimality of $\deg v$ since $w = (Eu)(Ev)^{-1}$. Therefore $Ev = 0$, which implies that $n = 0$ and so $v = c_0 \in U(H)$. It follows that $Eu = 0$ and $u \in U(H)$. Consequently, $w = uv^{-1} \in D(H)$ and thus $w \in Z(D(H))$. We have shown that $Z(D(L)) \subset Z(D(H))$. Moreover $Z(D(L)) \neq Z(D(H))$ since equality would imply $Z(U(L)) = Z(U(H))$ by (1).

(b) Suppose $Z(U(H)) \subsetneq Z(U(L))$. Then $Z(U(L)) \not\subset U(H)$ and $Z(U(H)) = Z(U(L)) \cap U(H)$. In other words, $U(L)^\Delta \not\subset U(H)$ and $Z(U(H)) = U(H)^\Delta$. Then the corollary and the remark following it assures the existence of an element $z = b_1 x + b_2 \in U(L)^\Delta = Z(U(L))$ with $b_1 \in U(H)^\Delta = Z(U(H))$, $b_1 \neq 0$, and $b_2 \in U(H)$, such that $D(H)[x]^\Delta = D(H)^\Delta[z]$ and $D(L)^\Delta = D(H)^\Delta(z)$ with z transcendental over $D(H)^\Delta$. Clearly $D(L)^\Delta = Z(D(L))$, so it remains to show that $D(H)^\Delta = Z(D(H))$. For this purpose we take $w \in Z(D(H))$. Then $0 = [w, z] = [w, b_1]x + b_1[w, x] + [w, b_2] = b_1[w, x]$. Hence $[w, x] = 0$ which implies that $w \in Z(D(L))$. So we have proved that $Z(D(H)) \subset Z(D(L))$. It follows that

$$Z(D(H)) = Z(D(L)) \cap D(H) = D(L)^\Delta \cap D(H) = D(H)^\Delta.$$

COROLLARY. *Let H be an ideal of L of codimension one such that $Z(U(H)) \neq Z(U(L))$. Then we have either $Z(D(H)) \subsetneq Z(D(L))$ or $Z(D(L)) \subsetneq Z(D(H))$.*

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