HYPERPLANARITY OF THE EQUIMULTIPLE LOCUS

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ABSTRACT. It is known that the (local) equimultiple locus of a hypersurface defined over a field of characteristic zero is contained in a hyperplane (for example, the one given by the Tschirnhausen transformation: see Abhyankar's paper [A] for details). In this note (a) we show that this theorem is no longer true for varieties of dimension bigger than two in char $p > 0$, and (b) we give proofs of this statement in the cases of

- (i) 'purely inseparable' surfaces,
- (ii) double points of surfaces in char 2.

1. Introduction. Let FA be a nonzero principal ideal in a regular local ring A . Assume A is excellent and equicharacteristic with infinite perfect residue field k . Let $p = \text{char } k$. Let

$$\mathcal{E}(A, F) = \{A_p \subset \text{qt.fld. of } A \mid P \in \text{Spec } A, \text{ord}_A F = \text{ord}_{A_p} F\}$$

denote the equimultiple locus of (A, F) . Let

$$\mathcal{E}^i(A, F) = \{S \in \mathcal{E}(A, F) \mid \dim S = i\}$$

denote the i -codimensional part of the equimultiple locus of (A, F) . Let

$$\text{equip}(A, F) = \{z \in M(A)^2 \mid Z \in M(S) \forall S \in \mathcal{E}(A, F)\}$$

denote the set of equimultiple parameters of (A, F) . Let

$$\text{equip}^i(A, F) = \{z \in M(A) \setminus M(A)^2 \mid Z \in M(S) \forall S \in \mathcal{E}^i(A, F)\}.$$

We say $\mathcal{E}(A, F)$ (resp. $\mathcal{E}^i(A, F)$) is hyperplanar if $\text{equip}(A, F)$ (resp. $\text{equip}^i(A, F)$) is nonempty.

The problem of hyperplanarity of the equimultiple locus of a hypersurface originated in the following observation: Assume $F = Z^d + a_1(X, Y)Z^{d-1} + \cdots + a_d$ with $a_i(X, Y) \in k[[X, Y]]$. Assume $\text{char } k = 0$ and $\text{ord}_A F = d$ where $A = k[[X, Y, Z]]$. Let P be a ht. 2 prime of A such that $F \in P^{(d)}$ (i.e., P is an equimultiple 'curve' for F). Then, taking $(d-1)$ derivatives with respect to Z , we get $Z + a_1(X, Y)/d \in P$, i.e., in geometric language, the equimultiple locus is contained in the hyperplane $(Z + a_1/d = 0)$. (For details and further discussion, especially about the 'infinitely near' equimultiple locus, see Abhyankar's forthcoming paper *Good points of a hypersurface* [A, §19].)

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In this paper we first give an example (§1) to show that the equimultiple locus of a hypersurface in char $p > 0$ is *not*, in general, hyperplanar.

On the positive side we prove the following:

Let $A = k[[X, Y, Z]]$, char $k = p > 0$.

THEOREM 1. Assume $F = Z^{p^n} + g(X, Y)$, $\text{ord}_A F = p^n$. Let P be a ht. 2 prime of A such that $\text{ord}_{A_P} F = p^n$. Then P contains a regular parameter of A . (In fact, $\text{equip}(A, F) \neq \emptyset$ (§3)).

THEOREM 2. Assume $p = 2$, $F = Z^2 + f(X, Y)Z + g(X, Y)$ with $f, g \in k[[X, Y]]$. Let P be a ht. 2 prime ideal of A such that $\text{ord}_{A_P} F = 2$. Then P contains a regular parameter of A (§3).

We first need a lemma that will allow us to use completions.

LEMMA. Let FA be a nonzero principal ideal in an excellent regular local domain A such that $\text{char } A = \text{char } A/M(A)$. Assume $\text{equip}(A^*, FA^*) \neq \emptyset$ where A^* is the completion of A . Then $\text{equip}(A, F) \neq \emptyset$.

PROOF. By hypothesis there exists $Z^* \in M(A^*) \setminus M(A^*)^2$ such that $Z^* \in M(S^*)$ for every $S^* \in \mathfrak{U}(A^*, FA^*)$.

Now by [N, 40.3] there exists a nonempty finite subset of $\mathfrak{U}(A, F)$, say $\{S_1, \dots, S_r\}$, such that

$$P = (A \cap M(S_1)) \cap \dots \cap (A \cap M(S_r)) \Rightarrow P \subset M(S)$$

for every $S \in \mathfrak{U}(A, F)$.

Hence

$$PA^* = (A \cap M(S_1))A^* \cap \dots \cap (A \cap M(S_r))A^*.$$

Also,

$$(A \cap M(S_i))A^* = (A^* \cap M(S_{i,1}^*)) \cap \dots \cap (A^* \cap M(S_{i,q(i)}^*))$$

where $\{S_{i,j}^*\}$ is a nonempty finite subset of $\mathfrak{U}(A^*, FA^*)$. Now,

$$\begin{aligned} Z^* \in PA^* &\Rightarrow \text{emdim } A^*/PA^* \leq (\dim A^*) - 1 \\ &\Rightarrow \text{emdim } A/P \leq (\dim A) - 1. \end{aligned}$$

Hence there exists $Z \in M(A) \setminus M(A)^2$ such that $Z \in P$. Since $P \subset M(S)$ for all $S \in \mathfrak{U}(A, F)$, we are done.

REMARK. In view of this Lemma we will restrict ourselves, in the following sections, to complete regular local rings A .

2. A counterexample in dimension > 2 . We now give an example to show that the equimultiple locus of a hypersurface of dimension ≥ 3 in characteristic p in general, is not hyperplanar.

EXAMPLE. Let $R = k[[X, Y, Z, W]]$, char $k = 2$. Let F be the purely inseparable polynomial (in W) given by $F = W^2 + YZ^3 + Y^3Z + X^7Y$. Note that $d = \text{ord}_R F = 2$. Let P be the kernel of the homomorphism

$$k[[X, Y, Z, W]] \xrightarrow{\phi} k[[T]] \quad (T \text{ an indeterminate})$$

defined by

$$X \rightarrow T^7, \quad Y \rightarrow T^{15}, \quad Z \rightarrow T^{19}, \quad W \rightarrow T^{32}.$$

Then $\dim R/P = 1$, and computing the singular locus of F , we get

$$F_X = Z^3 + X^6Y, \quad F_Y = Y^2Z + X^7, \quad F_Z = XZ^2 + Y^3.$$

Thus the definition of P shows that

$$F \in P, \quad F_X \in P, \quad F_Y \in P \quad \text{and} \quad F_Z \in P.$$

Hence, by the Jacobian criterion $F \in P^{(2)}$. Now, in the situation of the example, the following are equivalent:

(a) P contains a regular parameter of R .

(b) Among the integers $\text{ord}_T(\phi(X))$, $\text{ord}_T(\phi(Y))$, $\text{ord}_T(\phi(Z))$ and $\text{ord}_T(\phi(W))$, one is contained in the semigroup generated by the other three.

The proof is obvious and in the example, it only remains to check that condition (b) does not hold. Hence P is not hyperplanar.

3. The purely inseparable case. In view of the counterexample of the previous section, for the rest of the proofs we restrict our attention to *surfaces*.

Let $A = k[[X, Y, Z]]$. Assume $F = Z^{p^n} + g(X, Y)$, $\text{ord}_A F = p^n$, $\text{char } k = p > 0$, and $R = A/FA = k[[x, y, z]]$. Let P be a height two prime of A such that $F \in P^{(p^n)} = P^{p^n}A_P \cap A$. Let $Q = P \cap k[[X, Y]]$, $S = k[[X, Y]]/Q$. Then Q is a height one principal prime ideal in B , say $Q = (\phi)B$. We will first recall the proof for $n = 1$. The proof when $n > 1$ will be an extension of the basic techniques used for $n = 1$. The proof for $n = 1$ is due to Professor Abhyankar and may be found in [A]. For the convenience of the reader we reproduce the proof here.

THEOREM 1'. *Let $F = Z^p + g(X, Y)$. Assume $\text{ord}_A F = p$. Then, with the notation set up above, P contains a regular parameter of A .*

PROOF. Let $g(X, Y) = \sum_{0 \leq i \leq p-1, 0 \leq j \leq p-1} g_{ij} X^i Y^j$ be the canonical expression of $g(X, Y)$, with $g_{ij} \in k[[X, Y]]$.

Claim. Under the hypothesis of the theorem, $Z + g_{00}(X, Y) \in P$.

PROOF OF CLAIM. Let D_X, D_Y denote the usual partial derivatives with respect to X and Y .

Let x and y denote the residues of X and Y in S . Let $\bar{S}$ be the integral closure of S in its quotient field $k((x, y))$. Then $\bar{S} = k[[t]]$. Since $k((x, y)) = k((t)) \neq k((t^p))$, either x or y does not belong to $k((t^p))$, say $y \notin k((t^p))$. Then $\{1, y, y^2, \dots, y^{p-1}\}$ is a linearly independent set over $k((t^p))$.

Now

$$Z^p + g(X, Y) \in P^{(p)}, \quad D_X^i(g) \in P^{(p-i)}, \quad i = 1, 2, \dots, p-1.$$

For $i = p-1$ this gives $\sum_{j=0}^{p-1} g_{p-1,j} y^j = 0$ in S . By the linear independence of $\{y^j\}_{0 \leq j \leq p-1}$ we get $g_{p-1,j}(x, y) = 0$. Now by decreasing induction on i we get

$$g_{ij}(x, y) = 0 \quad \text{for } i = 1, 2, \dots, p-1, j = 0, 1, \dots, p-1.$$

But for each $j = 1, 2, \dots, p-1$ we have $D_Y^j(g) \in P^{p-j}$. Thus by decreasing induction on j we get

$$g_{ij}(x, y) = 0 \quad \text{for } (i, j) \neq (0, 0), 0 \leq i \leq p-1, 0 \leq j \leq p-1.$$

Thus $z^p + g_{00}^p(x, y) = 0$ in R i.e. $Z + g_{00}(X, Y) \in P$. This proves the claim and concretely exhibits the regular parameter of the theorem.

THEOREM 1. Let $G = Z^{p^n} + g(X, Y)$. Assume $F \in P^{(p^n)}$ and $\text{ord}_A F = p^n$. Then P contains a regular parameter of A .

PROOF. As before, let $g(X, Y)$ be expressed in terms of the usual p -basis $\{X^i Y^j\}_{0 \leq i < p, 0 \leq j < p}$ say

$$g = \delta^p + \sum_{\substack{(i,j) \neq (0,0) \\ i < p, j < p}} g_{ij}^p(X, Y) X^i Y^j.$$

Assume $y \notin k((t^p))$.

Then

$$(1) \quad D_X^i g = \alpha \sum_{0 \leq j < p} g_{ij}^p Y^j + \beta X^{k-i} \sum_{i < k < p} g_{kj}^p Y^j.$$

Claim. For $1 \leq i < p$, $g_{ij}^p \in P^{(p^n)}$ for $0 \leq j < p$.

PROOF. $F = Z^{p^n} + g \in P^{(p^n)}$

$$D_X^{(i)} g \in P^{(p^n-i)} \quad \text{for } i = 1, 2, \dots, p-1.$$

Hence $\alpha \sum_{0 \leq j < p} g_{ij}^p Y^j \in P^{(p^n-i)}$ from (1) above. (Note that $\alpha \not\equiv 0(p)$.) We prove the claim by decreasing induction on i . Let $\theta = \max \text{ord}_{S_0} g_{ij}$ for a fixed i .

Let $\lambda = \text{ord}_{R_p} \phi$. If $\theta = \infty$ then all the g_{ij} are zero in R . So we may assume $\theta < \infty$. Then $g_{ij} = \phi^\theta \gamma_{ij}$ with $\gamma_{ij} \in k[[X, Y]]$. For $i = p-1$ we have

$$\alpha \sum_{j=0}^{p-1} g_{i,j}^p Y^j \in P^{(p^n-(p-1))}.$$

If

$$\text{ord}_{R_p}(g_{ij}^p) = p\lambda\theta + \text{ord}_{R_p}(\gamma_{ij}^p) < p^n - (p-1),$$

then

$$\alpha \sum_{j=0}^{p-1} \gamma_{ij}^p \in P^{(p^n-\lambda\theta p-(p-1))}, \quad \alpha \sum_{j=0}^{p-1} \gamma_{ij}^p Y^j \in P.$$

Hence, by the case $n = 1$, $\gamma_{ij} \in P$, $i = p-1$, $j = 0, 1, \dots, p-1$. Thus $\gamma_{ij} \in Q = \phi | \gamma_{ij}$ contradicting the assumption that θ is the maximum power of ϕ which divides all the γ_{ij} . Thus we may assume $\text{ord}_{R_p}(g_{ij}^p) \geq p^n - (p-1)$. Hence $\text{ord}_{R_p}(g_{ij}^p) \geq p^n$ and this proves the claim for $i = p-1$. The rest of the claim follows by decreasing induction on i for $1 \leq i \leq p-1$.

PROOF OF THEOREM 1. By the claim we have $g_{ij}^p \in P^{(p^n)}$ for $1 \leq i \leq p-1$, $0 \leq j \leq p-1$. Now from $Z^{p^n} + g(X, Y) \in P^{(p^n)}$ taking derivatives with respect to

X , we get (by decreasing induction on j)

$$g_{ij}^p(X, Y) \in P^{(p^n)} \quad \text{for } (i, j) \neq (0, 0) \text{ and } 0 \leq i, j \leq p-1.$$

Hence $Z^{p^n} + g_{00}^p(X, Y) \in P^{(p^n)}$ i.e. $Z^{p^{(n-1)}} + g_{00}(X, Y) \in P^{(p^{n-1})}$. Now we are done by induction on n .

REMARK. The translate of Z , given by the *proofs* of Theorems 1' and 1 above, is *independent* of the equimultiple prime P i.e., every irreducible component of the equimultiple locus of F contains this regular parameter.

4. Double points of surfaces in characteristic two. As before we assume $A = k[[X, Y, Z]]$, $\text{char } k = 2$ and by the Weierstrass Preparation Theorem we assume $F = Z^2 + f(X, Y)Z + g(X, Y)$ with $f, g \in k[[X, Y]]$. Assume $\text{ord}_A F = 2$. Let P be a height 2 prime ideal of A such that $F \in P^{(2)}$.

THEOREM 2. $Z + p(X, Y) \in P$ for some $p(X, Y) \in k[[X, Y]]$.

PROOF. Let $B = k[[X, Y]]$, $Q = P \cap B$, $A/P = (R, m)$, $B/Q = (S, n)$. Let $\bar{S} = k[[t]]$, be the integral closure of S in its quotient field. Let x, y, z denote the residues of X, Y, Z modulo P . We will denote by D_X, D_Y and D_Z the usual partial derivatives with respect to X, Y and Z .

Now $F \in P^{(2)} \Rightarrow DF \in P$ for every derivation D of the quotient field of A . Hence $D_Z(F) = f \in P$; $D_X(F) = f_X Z + g_X \in P$; $D_Y(F) = f_Y Z + g_Y \in P$. Since $g(X, Y) \in k[[X, Y]]$, g can be written uniquely as

$$g(X, Y) = \delta^2 + \alpha^2 X + \beta^2 Y + \gamma^2 XY \quad \text{with } \alpha, \beta, \gamma, \delta \in B.$$

For the rest of the proof it will be convenient to write all our equations modulo P ; but we will continue to denote by α, β, γ and δ the residues of these quantities modulo P .

Case (i) (The purely inseparable case). I.e. both f_x and f_y are zero.

Then $\alpha^2 + \gamma^2 y = 0$, $\beta^2 + \gamma^2 x = 0$ in S , hence in $\bar{S} = k[[t]]$. Since $k((x, y)) = k((t))$, either $x \notin k((t^2))$ or $y \notin k((t^2))$.

Hence $\gamma = 0 \Rightarrow \alpha = \beta = 0 \Rightarrow z = 0 \Rightarrow Z \in P$, proving the claim.

Case (ii) (The separable cyclic case). We may assume either f_x or f_y is not zero, say $f_x \neq 0$. Then $f_x z + g_x = 0 \Rightarrow R$ and S are birational and $F = 0 \Rightarrow R$ is integral over S . Hence $S \subseteq R \subseteq \bar{S} = k[[t]]$. Let $\mathfrak{C}$ denote the conductor of $\bar{S}/S$. We have

$$(1) \quad z^2 + \delta^2 + \alpha^2 x + \beta^2 y + \gamma^2 xy = 0,$$

$$(2) \quad f_x z + \alpha^2 + \gamma^2 y = 0, \quad \dots$$

$$(3) \quad f_y z + \beta^2 + \gamma^2 x = 0.$$

Now,

$$f(X, Y) \in P \Rightarrow f(X, Y) \in Q \Rightarrow \phi(X, Y) | f(X, Y).$$

Hence D_X modulo P give $\phi_x | f_x \Rightarrow \text{ord}_t f_x \geq \text{ord}_t \phi_x$.

From equations (2) and (3) we get

$$(xf_x + yf_y)z + \alpha^2 x + \beta^2 y = 0.$$

Since $\text{ord}_t(f_x) \geq \text{ord}_t \mathfrak{C} = c$, we have

$$(4) \quad \text{ord}_t(\alpha^2 x + \beta^2 y) \geq \text{ord}_t z + c.$$

From equation (2), since $\text{ord}_t \gamma^2 + \text{ord}_t(dy) + 1$ is odd and $\text{ord}_t(\alpha^2)$ is even we have

$$\text{ord}_t \gamma^2 + \text{ord}_t(dy) + 1 \geq \text{ord}_t(f_x) + \text{ord}_t z.$$

Now by the Dedekind conductor formula we have $\text{ord}_t(\phi_x) = \text{ord}_t(dy) + c$. Hence

$$(5) \quad \text{ord}_t(\gamma^2) + 1 \geq c + \text{ord}_t z.$$

Now, in equation (1),

(a) if $\text{ord}_t(z^2) \geq \text{ord}_t(\alpha^2 x + \beta^2 y)$ then $\text{ord}_t z \geq c$ by (4),

(b) if $\text{ord}_t(z^2) \geq \text{ord}_t(\gamma^2 xy)$ then $\text{ord}_t z \geq c$ by (5),

(c) if neither (a) nor (b) holds then

$$\text{ord}_t(z^2) = \text{ord}_t(\delta^2) \quad \text{and} \quad \text{ord}_t(z + \delta) \neq \text{ord}_t(z).$$

Hence repeating the argument with z replaced by $z + \delta$, after a finite number of steps we get $\text{ord}_t(z + h(x, y)) \geq c$ for some $h(X, Y) \in k[[X, Y]]$. In all three cases we thus have $z + h(x, y) \in S$ for some h , hence $z + p(x, y) = 0$ in S , for some $p(X, Y) \in k[[X, Y]]$ i.e. $Z + p(X, Y) \in P$.

This proves the theorem.

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