RECOVERABILITY OF SOME CLASSES OF ANALYTIC FUNCTIONS FROM THEIR BOUNDARY VALUES*

AHMED I. ZAYED

ABSTRACT. The technique devised by D. J. Patil to recover the functions of the Hardy space H^p ($1 \leq p \leq \infty$) from the restrictions of their boundary values to a set of positive measure on the unit circle was modified by S. E. Zarantonello in order to extend the result to H^p ($0 < p < 1$).

In this paper, we show that Zarantonello's technique can be slightly modified to extend the result to a larger class of analytic functions in the unit disc. In particular, if $f(z)$ is analytic in the unit disc and satisfies

$$\lim_{r \rightarrow 1} (1-r)^\beta \log M(r, f) = 0 \quad \text{for some } \beta \geq 1,$$

then $f(z)$ can be recovered from the restriction of its boundary value to an open arc.

1. Introduction. Let $\mathbb{D}$ denote the open unit disc, $\partial\mathbb{D}$ its boundary, i.e., the unit circle, and μ the normalized Lebesgue measure on $\partial\mathbb{D}$. Furthermore, let H^p ($0 < p \leq \infty$) denote the Hardy class of analytic functions in $\mathbb{D}$. It is well known that if $f(z) \in H^p$ and $\lim_{r \rightarrow 1} f(re^{i\theta}) = 0$ on a set $E \subset \partial\mathbb{D}$ of positive measure, then $f(z)$ is identically zero. We call this property the uniqueness property. In a sense, this means that functions in H^p are uniquely determined by their values on E .

The question of whether a function $f \in H^p$ can be recovered from its restriction to E was answered in the affirmative by D. Patil [1] for $1 \leq p \leq \infty$, who constructed a sequence of analytic functions that converged to f uniformly on compact subsets of $\mathbb{D}$ as well as in the norm. Modifying Patil's technique, S. Zarantonello [4] and G. Walker [3] independently were able to extend the results to $0 < p < 1$ but in a slightly more restrictive form.

A function $f(z) \in H^p$ ($0 < p < 1$) can be recovered from the restriction of its *distributional boundary value* to E where E now is an open arc in $\partial\mathbb{D}$.

Since there are larger classes of analytic functions in $\mathbb{D}$ having the uniqueness property, e.g., the Nevanlinna class N , a natural question immediately arises, can functions in these classes be recovered from their restrictions to E ? The purpose of this article is to show that the answer is "yes" for a large class of analytic functions. More precisely, if we denote by $\mathcal{H}(\alpha)$ ($0 < \alpha < 1$) the class of all analytic functions in $\mathbb{D}$ having the property $\int_0^1 M(r, f) \exp(-c/(1-r)^\beta) dr < \infty$ for all $c > 0$ where

This paper was presented at the 791st meeting of the American Mathematical Society in Santa Barbara, California, November 1981; received by the editors October 5, 1981.

1980 *Mathematics Subject Classification.* Primary 30E25; Secondary 30B30.

Key words and phrases. Recoverability theorem, Beurling distributions, Toeplitz operators.

*This paper previously appeared in Volume 86 of the Proceedings on pages 97–102. It is being reprinted because the pages appeared out of sequence.

$M(r, f) = \sup_{|z|=r} |f(z)|$ and $\beta = \alpha/(1 - \alpha)$, then any function $f(z) \in \mathcal{H}(1) = \bigcup_{0 < \alpha < 1} \mathcal{H}(\alpha)$ can be recovered from the restriction of its distributional boundary value to an open arc E . The technique we use is a modification of the ones given by Patil and Zarantonello.

2. Preliminaries. For $0 < \alpha < 1$, let $\mathcal{H}(\alpha)$ be the space of all holomorphic functions $F(z)$ in $\mathbb{D}$ such that $F(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n = O(e^{o(n^\alpha)})$ as $n \rightarrow \infty$. The topology of $\mathcal{H}(\alpha)$ is the topology induced by the seminorms

$$(2.1) \quad \|F\|_c = \sum_{n=0}^{\infty} |a_n| e^{-cn^\alpha} < \infty \quad \text{for } c > 0.$$

In [5 and 6], we showed the following facts.

(1) $F(z) \in \mathcal{H}(\alpha)$ if and only if

$$\|F\|_c = \int_0^1 M(r, f) \exp \frac{-c}{(1-r)^\beta} dr < \infty \quad \text{for all } c > 0.$$

Moreover, the two families of seminorms $\|\cdot\|_c$ and $\|\cdot\|_c$ are equivalent.

(2) Provided with that topology, $\mathcal{H}(\alpha)$ becomes a Fréchet-Montel space whose topology is stronger than the topology of uniform convergence on compact subsets of $\mathbb{D}$.

(3) If we denote by $\mathcal{Q}_\alpha(\mathbb{D})$ the algebra of functions $G(z)$ that are analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ such that $G(z) = \sum_{n=0}^{\infty} b_n z^n$, $b_n = O(e^{-cn^\alpha})$ as $n \rightarrow \infty$ for some $c > 0$, then for any $\phi \in \mathcal{H}^*(\alpha)$ (the dual space of $\mathcal{H}(\alpha)$), there exists a unique function $G(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{Q}_\alpha(\mathbb{D})$ such that

$$(2.2) \quad \phi(F) = \lim_{r \rightarrow 1} \int_{\partial \mathbb{D}} F(re^{i\theta}) G(e^{-i\theta}) d\mu = \sum_{n=0}^{\infty} a_n b_n.$$

Conversely, any $G(z) \in \mathcal{Q}_\alpha(\mathbb{D})$ defines a continuous linear functional on $\mathcal{H}(\alpha)$ via (2.2).

Let $\mathcal{Q}_\alpha = \mathcal{Q}_\alpha(\partial \mathbb{D})$ be the space of all C^∞ -functions ϕ on $\partial \mathbb{D}$ such that

$$\phi(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$$

with $b_n = O(e^{-c|n|^\alpha})$ as $n \rightarrow \infty$ for some $c > 0$.

Another useful characterization of the class $\mathcal{Q}_\alpha$ is that $\phi \in \mathcal{Q}_\alpha$ if and only if $\sup_{0 \leq \theta \leq 2\pi} |\phi^{(n)}(\theta)| \leq KR^{n/\alpha} n^{n/\alpha}$ for some constants K and R .

Indeed, the class $\mathcal{Q}_\alpha$ is the same as the class $C\{(n!)^{1/\alpha}\}$ in the notation of [2].

It follows from the Denjoy-Carleman theorem that $\mathcal{Q}_\alpha$ is a nonquasianalytic class of functions on $\partial \mathbb{D}$. A topology is defined on $\mathcal{Q}_\alpha$ by means of the fundamental system of neighborhoods of the origin

$$V(\lambda) = \left\{ \phi \mid \phi(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, b_n = O(e^{-\lambda_n |n|^\alpha}) \right\}$$

where $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ and $\lambda_n \downarrow 0$.

The strong dual $\mathcal{Q}_\alpha^*$ of $\mathcal{Q}_\alpha$ is a space of Beurling distributions which are more general than Schwartz distributions (see [5 and 6] for references). One can verify the following.

(1) Every Beurling distribution f has a Fourier series expansion $f = \sum_{-\infty}^{\infty} a_n w^n$ that converges weakly to it, where $w = e^{i\theta}$ and $a_n = \langle f, w^n \rangle$.

(2) A necessary and sufficient condition that $f = \sum_{-\infty}^{\infty} a_n w^n \in \mathfrak{A}_\alpha^*$ is that $a_n = O(e^{\alpha(|n|^\alpha)})$ as $n \rightarrow \infty$. Furthermore, if $\psi = \sum_{-\infty}^{\infty} b_n w^n \in \mathfrak{A}_\alpha^*$ then $\langle f, \psi \rangle = \sum_{-\infty}^{\infty} a_n \bar{b}_n$.

(3) If $F(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\alpha)$, then $f = \sum_{n=0}^{\infty} a_n w^n$ is a Beurling distribution belonging to the space $\mathfrak{A}_\alpha^*$. In addition to that (cf. [5, Corollary 3.1])

$$(2.3) \quad F_r(w) = F(rw) \rightarrow f \quad \text{in } \mathfrak{A}_\alpha^*.$$

As in [4], we say that f is the distributional boundary value of $F(z)$ and $F(z)$ is the holomorphic extension of f . The space of all distributional boundary values of functions in $\mathcal{H}(\alpha)$ will be denoted by $\tilde{\mathfrak{A}}_\alpha^*$. We provide $\tilde{\mathfrak{A}}_\alpha^*$ with a topological structure isometric to that of $\mathcal{H}(\alpha)$ by setting

$$(2.4) \quad \|f\|_c = \|F\|_c.$$

From (2.1) and (2.2), one can easily see that the topology of $\tilde{\mathfrak{A}}_\alpha^*$ is stronger than the topology it inherits from $\mathfrak{A}_\alpha^*$.

From Cauchy's formula

$$F(rz) = \int_{\partial D} \frac{F_r(w)}{1 - \bar{w}z} d\mu(w)$$

and equation (2.3), it follows that $F(z) = \langle f, C_z \rangle$ where $C_z(w) = 1/(1 - \bar{w}z)$.

3. Toeplitz operators on $\tilde{\mathfrak{A}}_\alpha^*$. Since $\mathfrak{A}_\alpha$ is an algebra [2], we define the multiplication operator M_ϕ for $\phi \in \mathfrak{A}_\alpha$ by

$$\langle M_\phi f, \psi \rangle = \langle f, \bar{\phi} \psi \rangle$$

where $f \in \mathfrak{A}_\alpha^*$ and $\psi \in \mathfrak{A}_\alpha$.

The projection operator $P: \mathfrak{A}_\alpha^* \rightarrow \tilde{\mathfrak{A}}_\alpha^*$ is formally given by

$$P\left(\sum_{-\infty}^{\infty} a_n w^n\right) = \sum_0^{\infty} a_n w^n$$

and hence the Toeplitz operator S_ϕ can now be defined on $\tilde{\mathfrak{A}}_\alpha^*$ as $S_\phi = PM_\phi$ for $\phi \in \mathfrak{A}_\alpha$.

LEMMA 1. Let $\phi \in \mathfrak{A}_\alpha$ and $f \in \tilde{\mathfrak{A}}_\alpha^*$, then:

(i) For any $c > 0$ there exists $\tilde{c}$ such that

$$\|S_\phi f\|_c \leq K(\phi, c) \|f\|_{\tilde{c}}$$

where $K(\phi, c)$ is a constant that depends only on c and ϕ but not on f .

(ii) $S_\phi f$ is the distributional boundary value of the analytic function $\langle M_\phi f, C_z \rangle$.

PROOF. (i) Let $f = \sum_0^{\infty} a_n w^n$ and $\phi = \sum_{-\infty}^{\infty} b_n w^n$, then $S_\phi f = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_n \bar{b}_{k-n} w^k$. Hence,

(3.1)

$$\begin{aligned} \|S_\phi f\|_c &\leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_n| |b_{k-n}| e^{-c|k|^\alpha} = \sum_{n=0}^{\infty} |a_n| \sum_{k=-n}^{\infty} |b_k| e^{-c|k+n|^\alpha} \\ &= \sum_{n=0}^{\infty} |a_n| \sum_{k=-n}^{-1} |b_k| e^{-c|k+n|^\alpha} + \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{\infty} |b_k| e^{-c|k+n|^\alpha} = I_1 + I_2. \end{aligned}$$

But $(k+n)^\alpha \leq k^\alpha + n^\alpha \leq 2(k+n)^\alpha$, therefore

$$(3.2) \quad I_2 \leq \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{\infty} |b_k| \exp\left(-\frac{1}{2}c|k|^\alpha - \frac{1}{2}c|n|^\alpha\right) \leq B\|f\|_{c/2}$$

where $B = \sum_{k=0}^{\infty} |b_k| \exp(-\frac{1}{2}c|k|^\alpha)$ which certainly converges since $|b_k| \leq Ae^{-b|k|^\alpha}$ for some $b > 0$. On the other hand,

$$(3.3) \quad \begin{aligned} I_1 &\leq \sum_{n=0}^{\infty} |a_n| \sum_{k=-n}^{-1} Ae^{-b|k|^\alpha - c|k+n|^\alpha} \\ &\leq A \sum_{n=0}^{\infty} |a_n| ne^{-2d|n|^\alpha} \leq \tilde{B} \sum_{n=0}^{\infty} |a_n| e^{-d|n|^\alpha} = \tilde{B}\|f\|_d \end{aligned}$$

where $d = \frac{1}{2} \min(b, c)$ and $\tilde{B} = \max_{0 \leq n} A(n/e^{dn^\alpha}) < \infty$.

From (3.1), (3.2) and (3.3), it follows that $\|S_\phi f\|_c \leq K\|f\|_{\tilde{c}}$ where $K = \max(B, \tilde{B})$ and $\tilde{c} = \min(c/2, d)$.

(ii) Let $f \in \mathcal{H}(\alpha)$ and $F(z)$ be its holomorphic extension to $\mathcal{V}$. From Corollaries 3.1 and 3.2 of [5], it follows that $F_r(w) = F(rw) \rightarrow f$ in $\mathcal{H}(\alpha)$ and that H^2 is dense in $\mathcal{H}(\alpha)$. Moreover, $F_r(w)$ is the boundary value of a function in H^2 , namely $F_r(z)$. Now an argument similar to that of Theorem 3.2 of [4] finishes the proof.

4. Recoverability Theorem. In this section we shall use the same notation as [4]. Let E be an open arc in $\partial\mathcal{V}$ and $\psi \in \mathcal{Q}_\alpha$ such that $0 \leq \psi \leq 1$, $\text{supp } \psi \subset E$ and $\mu(J) > 0$ where $J = \{w \in \partial\mathcal{V} : \psi(w) = 1\}$. Such a ψ certainly exists since the class $\mathcal{Q}_\alpha$ is nonquasianalytic. For each $0 < \lambda < 1$ define

$$\phi_\lambda(w) = \frac{1}{1 - \lambda\psi(w)}$$

and

$$H_\lambda(z) = \exp\left\{-\frac{1}{2} \int_{\partial\mathcal{V}} \frac{w+z}{w-z} \log \phi_\lambda(w) d\mu\right\}, \quad z \in \mathcal{V}.$$

As is shown in [1 and 4], if we denote the boundary value of $H_\lambda(z)$ by $h_\lambda(w)$, we have

- (a) $|h_\lambda(w)|^{-2} = \phi_\lambda(w)$.
- (b) h_λ and h_λ^{-1} are in $H^\infty(\partial\mathcal{V})$.

LEMMA 2. (i) $\phi_\lambda \in \mathcal{Q}_\alpha$.

(ii) $S_{\phi_\lambda} : \tilde{\mathcal{Q}}_\alpha^* \rightarrow \tilde{\mathcal{Q}}_\alpha^*$ is invertible with inverse $S_{\phi_\lambda}^{-1} = S_{h_\lambda} S_{h_\lambda}^{-1}$.

(iii) For any $c > 0$, there exists $\tilde{c}$ such that

$$\|S_{\phi_\lambda}^{-1} f\|_c \leq K\|f\|_{\tilde{c}} \quad \text{where } K \text{ is independent of } \lambda.$$

PROOF. (i) Since $\psi \in \mathcal{Q}_\alpha$ and $\mathcal{Q}_\alpha$ is an inverse-closed, nonquasianalytic class of functions [2], it follows immediately that $\phi_\lambda \in \mathcal{Q}_\alpha$.

(ii) It suffices to show that $h_\lambda \in \mathcal{Q}_\alpha$. Let $h_\lambda(w) = e^{-(u(w) + iv(w))}$. Since $\log \phi_\lambda(w) \in \mathcal{Q}_\alpha$ by part (i) and Theorem A of [2], it is easy to see that $u(w)$ and $v(w)$ are also in $\mathcal{Q}_\alpha$. We invoke Theorem A once more to show that h_λ is indeed in $\mathcal{Q}_\alpha$.

(iii) $\|S_{h_\lambda} S_{h_\lambda}^{-1} f\|_c \leq \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\hat{h}_\lambda(k)| |\hat{h}_\lambda(m)| |\hat{f}(n)| e^{-c|k-m+n|^\alpha}$.

Since $|h_\lambda(w)| \leq (\phi_\lambda(w))^{-1/2} \leq 1$, it follows that the sequence $\{\hat{h}_\lambda(k)\}$ is uniformly bounded. Now an argument similar to the one given in Lemma 1 yields the result.

LEMMA 3. For fixed $z \in \mathcal{D}$ and $c > 0$, $\lim_{\lambda \rightarrow 1} \|S_{\phi_\lambda}^{-1} C_z\|_c = 0$.

PROOF. The relation (cf. [4, equation (4.4.2)])

$$|H_\lambda(z)| = \exp \left\{ -\frac{1}{2} \int \frac{1 - |z|^2}{|1 - \bar{w}z|^2} \log \phi_\lambda(w) d\mu(w) \right\} \leq (1 - \lambda)^\alpha$$

where $2\alpha = \{(1 - |z|)/(1 + |z|)\}\mu(J) > 0$, shows that $H_\lambda(z) \rightarrow 0$ uniformly on compact subsets of $\mathcal{D}$ as $\lambda \rightarrow 1$. Since $H_\lambda(z) = \sum_{k=0}^{\infty} \hat{h}_\lambda(k) z^k$, it is routine to show that $\hat{h}_\lambda(k) \rightarrow 0$ as $\lambda \rightarrow 1$ for $k = 0, 1, 2, \dots$. Moreover, $|\hat{h}_\lambda(k)| \leq B$ for all $0 < \lambda < 1$ and $k = 0, 1, 2, \dots$.

We claim that $H_\lambda(z) \rightarrow 0$ in $\mathcal{H}(\alpha)$ as $\lambda \rightarrow 1$. For

$$(4.1) \quad \|H_\lambda\|_c = \sum_{k=0}^{\infty} |\hat{h}_\lambda(k)| e^{-ck^\alpha} = \sum_{k=0}^N |\hat{h}_\lambda(k)| e^{-ck^\alpha} + \sum_{N+1}^{\infty} |\hat{h}_\lambda(k)| e^{-ck^\alpha}.$$

Choose N large enough so that $\sum_{N+1}^{\infty} e^{-ck^\alpha} \leq \varepsilon/2B$ and λ so close to 1 that $|\hat{h}_\lambda(k)| \leq \varepsilon/2(N+1)$ for $k = 0, 1, \dots, N$. Substitution in (4.1) gives

$$(4.2) \quad \|H_\lambda\|_c \leq \varepsilon \quad \text{for } \lambda \text{ sufficiently close to 1.}$$

By an argument similar to the one given in Lemma 1, the fact that $S_{\phi_\lambda}^{-1} C_z = \bar{H}_\lambda(z) h_\lambda C_z$ and (4.2) one easily obtains

$$\lim_{\lambda \rightarrow 1} \|S_{\phi_\lambda}^{-1} C_z\|_c = \lim_{\lambda \rightarrow 1} \|\bar{H}_\lambda(z) h_\lambda C_z\|_c \leq A \lim_{\lambda \rightarrow 1} \|H_\lambda\|_{\tilde{c}} = 0$$

for some $\tilde{c} > 0$ and a constant A that depends on z and c but not on λ .

LEMMA 4. Let $f \in \tilde{\mathcal{Q}}_\alpha^*$ and $g_\lambda = S_{\phi_\lambda}^{-1}(S_{\phi_\lambda} - I)f$, then $\lim_{\lambda \rightarrow 1} g_\lambda = f$ in $\tilde{\mathcal{Q}}_\alpha^*$.

PROOF. As a consequence of Corollary 3.2 of [5], $L^\infty(\partial\mathcal{D})$ is dense in $\tilde{\mathcal{Q}}_\alpha^*$. From this and the fact that $\{C_z: z \in \mathcal{D}\}$ is a fundamental set in $L^2(\partial\mathcal{D})$ and that the embedding $L^2(\partial\mathcal{D}) \rightarrow \tilde{\mathcal{Q}}_\alpha^*$ is continuous, we conclude that $\{C_z: z \in \mathcal{D}\}$ is also a fundamental set in $\tilde{\mathcal{Q}}_\alpha^*$. This fact together with Lemmas 2 and 3 may now be used to show that $\lim_{\lambda \rightarrow 1} \|S_{\phi_\lambda}^{-1} f\|_c = 0$. But $S_{\phi_\lambda}^{-1} f = f - g_\lambda$ which yields the result.

Using arguments parallel to those given in [4 and 1], the reader should be able to finish the proof of the following theorem.

THEOREM. Let E be an open arc of $\partial\mathcal{D}$. Let $F(z) \in \mathcal{H}(\alpha)$ and f be its distributional boundary value on $\partial\mathcal{D}$. Suppose that g is the restriction of f to E . For each $0 < \lambda < 1$ define an analytic function $G_\lambda(z)$ on $\mathcal{D}$ by

$$G_\lambda(z) = H_\lambda(z) \langle g, (\phi_\lambda - 1) h_\lambda C_z \rangle.$$

Then, $\lim_{\lambda \rightarrow 1} G_\lambda(z) = F(z)$ in $\mathcal{H}(\alpha)$. In particular, $\lim_{\lambda \rightarrow 1} G(z) = F(z)$ uniformly on compact subsets of $\mathcal{D}$.

It is well known that functions in $\mathcal{H}(\alpha)$ can have radial limits equal to zero on a set E of positive measure (in fact, even on a set of measure 2π) without being

identically zero i.e., the class $\mathcal{H}(\alpha)$ does not have the uniqueness property. However, the class $\mathcal{H}(\alpha)$ possesses another uniqueness property which we state as a corollary.

COROLLARY. *Let $F(z) \in \mathcal{H}(\alpha)$ and f be its distributional boundary value. If $f = 0$ (in the sense of distributions) on an open arc E , then $F(z)$ is identically zero.*

REMARKS. (i) One may ask how far can the result be extended? To answer this question, let us consider the class $\mathcal{H}(\omega)$ of all analytic functions $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{D}$ with $a_n = O(e^{o(\omega(n))})$ as $n \rightarrow \infty$ where $\omega(x)$ is continuously differentiable and monotonically increasing on $[0, \infty)$. Certainly, our technique fails if $\sum_{n=0}^{\infty} \omega(n)/n^2 = \infty$, since in this case the class $\mathcal{G}_\omega$ is quasianalytic by the Denjoy-Carleman theorem. Therefore, if the function $\psi(w) = 0$ on a set of positive measure, it is identically zero on $\partial\mathcal{D}$ and hence $G_\lambda(z)$ is identically zero for all $0 < \lambda < 1$.

(ii) The class N^+ provided with the topology induced by the metric

$$(f, g) = \int_{\partial\mathcal{D}} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) d\mu$$

where f and $g \in N^+$ is a subspace of $\mathcal{H}(1/2)$.

From the main theorem, it follows that if $F \in N^+$, then $G_\lambda(z) \rightarrow F(z)$ in $\mathcal{H}(1/2)$. It would be interesting to know whether $G_\lambda(z) \rightarrow F(z)$ in N^+ .

ACKNOWLEDGEMENT. The author wishes to thank Professor Walter Rudin for bringing to his attention the results of reference [2].

REFERENCES

1. D. Patil, *Representation of H^p -functions*, Bull. Amer. Math. Soc. **4** (1972), 617–620.
2. W. Rudin, *Division in algebras of infinitely differentiable functions*, J. Math. Mech. **11** (1962), 797–810.
3. G. Walker, *Analytic representations, values, and recoverability of distributions*, Ph.D. thesis, University of Wisconsin, Milwaukee, 1974.
4. S. Zarantonello, *A representation of H^p functions with $0 < p < \infty$* , Pacific J. Math. **79** (1978), 271–282.
5. A. Zayed, *Topological vector spaces of analytic functions*, submitted.
6. ———, *On Beurling distributions and holomorphic functions in the unit disc*, submitted.

DEPARTMENT OF MATHEMATICS, CALIFORNIA POLYTECHNIC STATE UNIVERSITY, SAN LUIS OBISPO, CALIFORNIA 93407