ON SUPERCOMPLETE UNIFORM SPACES

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ABSTRACT. We show that a uniform space μX is supercomplete if, and only if, the Ginsburg-Isbell locally fine coreflection of $\mu X \times \mathfrak{F}\beta X$ is equinormal.

1. Introduction. J. Isbell calls a uniform space μX supercomplete if the uniform hyperspace $H(\mu X)$ of closed subsets of X is complete. He proved in [4] that μX is supercomplete if, and only if, X is paracompact and $\lambda\mu = \mathfrak{F}_X$, where $\mathfrak{F}_X$ is the fine uniformity of X . (On the other hand, Morita proved in [7] that the uniform hyperspace $K(\mu X)$ of all compact subsets of X is complete if and only if μX is complete.) Tamano established in 1960 (see [9]) the following famous result: a completely regular space X is paracompact whenever $X \times \beta X$ is normal. It seems reasonable to ask if it is possible to give a Tamano-like “finitary” characterization of supercomplete uniform spaces by means of their Čech-Stone compactifications. First of all, we will recall some familiar definitions. A quasiuniformity is a filter of coverings ordered by the relation of refinement. If μ and ν are quasiuniformities, then μ/ν is the quasiuniformity consisting of all covers $\{V_i \cap W_j^i\}$, where $\{V_i\}_i \in \nu$ and for each i , $\{W_j^i\}_j \in \mu$. If μ is a quasiuniformity, let $\mu^{(0)} = \mu$. If $\mu^{(\alpha)}$ is defined for all $\alpha < \beta$ and β is a limit ordinal, let $\mu^{(\beta)} = \bigcup \{\mu^{(\alpha)} : \alpha < \beta\}$. On the other hand, if $\beta = \alpha + 1$, then we define $\mu^{(\beta)} = \mu^{(\alpha)}/\mu^{(\alpha)}$. There is an ordinal α such that $\mu^{(\alpha)} = \mu^{(\alpha+1)}$ and for which we define $\lambda\mu = \mu^{(\alpha)}$. In case μ is a uniformity, it can be shown that [1, Theorem 4.4] $\lambda\mu$ is the coarsest locally fine [1] uniformity finer than μ . The operator λ is called the Ginsburg-Isbell locally fine coreflection. The precompact reflection of μ will be denoted by $p\mu$. A uniform space is equinormal if every binary open cover of the space is uniform. The symbol $\mathfrak{V}|A$ (resp. $\mu|A$) denotes the trace of the family $\mathfrak{V}$ on A (resp. the induced quasiuniformity of A).

2. The result. Tamano proved the following lemma in [9].

LEMMA 1. *A completely regular space X is paracompact if and only if for each compact $K \subset \beta X \sim X$ there is a locally finite open cover $\mathfrak{V}$ of X such that $\text{cl}_{\beta X} V \cap K = \emptyset$ for all $V \in \mathfrak{V}$.*

We will extend Lemma 1 to “ λ -uniform” covers. ..

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LEMMA 2. *A uniform space μX is supercomplete if and only if for each compact $K \subset \beta X \sim X$ there is a $\mathfrak{V} \in \lambda\mu$ such that $\text{cl}_{\beta X} V \cap K = \emptyset$ for all $V \in \mathfrak{V}$.*

PROOF. To prove sufficiency, suppose that the condition given in Lemma 2 holds. By Lemma 1, X is paracompact since each element of $\lambda\mu$ has a locally finite open refinement [5, I.14]. We will show that $\lambda\mu = \mathfrak{T}_X$. To this end, let $\mathfrak{V}$ be an open cover of X . For each element V of $\mathfrak{V}$ choose an open set V^* in βX with $V = V^* \cap X$. Then $K = \beta X \sim \bigcup \{V^*: V \in \mathfrak{V}\}$ is compact. Hence, there exists a $\mathfrak{W} \in \lambda\mu$ such that $\text{cl}_{\beta X} W \cap K = \emptyset$ for all $W \in \mathfrak{W}$. It is not difficult to see that $\mathfrak{W}$ refines the cover $\mathfrak{V}^{<\omega}$ consisting of all finite unions of sets from $\mathfrak{V}$. As $\mathfrak{V}$ is arbitrary, Theorems 2 and 3 in [8] show that $\mathfrak{T}_X = p\lambda\mu/\lambda\mu$. But $p\lambda\mu/\lambda\mu = \lambda\mu$. Hence μX is supercomplete.

On the other hand, let μX be supercomplete and assume that $K \subset \beta X \sim X$ is compact. By Isbell's result X is paracompact and $\lambda\mu = \mathfrak{T}_X$. Let $\mathfrak{V}$ be as in Lemma 1. Since X is paracompact, we have $\mathfrak{V} \in \mathfrak{T}_X = \lambda\mu$, as desired.

To prove the main result, we shall need an auxiliary derivative introduced in [3]. Let $\mu_0^{(0)} = \mu$. If $\mu_0^{(\alpha)}$ is defined for all $\alpha < \beta$ and β is a limit ordinal, let $\mu_0^{(\beta)} = \bigcup \{\mu_0^{(\alpha)}: \alpha < \beta\}$; if $\beta = \alpha + 1$, then we define $\mu_0^{(\beta)} = \mu_0^{(\alpha)}/\mu$. The following basic lemma can be found in [3]. The statement is proved by a transfinite induction argument.

LEMMA 3. *For each α there is a $\sigma(\alpha)$ such that $\mu^{(\alpha)} \subset \mu_0^{(\sigma(\alpha))}$.*

It follows that there is an ordinal α with $\mu_0^{(\alpha)} = \lambda\mu$.

Finally, we shall need the following result proved in [2].

LEMMA 4. *Let μX be a uniform space and let K be a compact space. Then $(\mathfrak{T}_X \times \mathfrak{T}_K)^{(1)} = \mathfrak{T}_{(X \times K)}$.*

Note that in the preceding lemma both $(\mathfrak{T}_X \times \mathfrak{T}_K)^{(1)}$ and $\mathfrak{T}_{(X \times K)}$ are uniformities on $X \times K$.

THEOREM. *A uniform space μX is supercomplete if, and only if, the Ginsburg-Isbell locally fine coreflection of $\mu X \times \mathfrak{T}\beta X$ is equinormal.*

PROOF. To prove sufficiency, we will proceed by transfinite induction. Let $K \subset \beta X \sim X$ be compact. Then $X \times K$ and $\Delta(X)$ (the diagonal of $X \times X$) are closed and disjoint subsets of $X \times \beta X$. Define $G_1 = (X \times \beta X) \sim (X \times K)$ and $G_2 = (X \times \beta X) \sim \Delta(X)$. Then $\mathfrak{G} = \{G_1, G_2\}$ is a binary open cover of $X \times \beta X$. Let $P(\alpha)$ be the following statement: if νY is a uniform space, $K \subset \beta Y \sim Y$ is compact, $\mathfrak{G}$ is defined as above, and $\mathfrak{G} \in (\nu \times \mathfrak{T}_{\beta Y})_0^{(\alpha)}$, then there is a $\mathfrak{V} \in \nu_0^{(\alpha)}$ such that $\text{cl}_{\beta Y} V \cap K = \emptyset$ for all $V \in \mathfrak{V}$. Theorem 5.1.2 in [2] shows that $P(\alpha)$ is valid for $\alpha = 1$. For the induction hypothesis, suppose that $P(\alpha)$ holds whenever $1 \leq \alpha < \tau$. Let $\mathfrak{G} \in (\mu \times \mathfrak{T}_{\beta X})_0^{(\tau)}$. If τ is a limit ordinal, then the induction hypothesis implies directly that there is a $\mathfrak{V} \in \mu_0^{(\tau)}$ with the desired property. Thus, we can assume that $\tau = \alpha + 1$. There is a uniform cover $\mathfrak{U}$ of $X \times \beta X$ such that for each $U \in \mathfrak{U}$ we have $\mathfrak{G} \upharpoonright U \in [(\mu \times \mathfrak{T}_{\beta X}) \upharpoonright U]_0^{(\alpha)}$. Choose $\{U_i\} \in \mu$ and $\{V_j\} \in \mathfrak{T}_{\beta X}$ such that $\{U_i \times V_j\} < \mathfrak{U}$. As βX is compact we can assume that $\{V_j\} = \{V_1, \dots, V_n\}$. We can assume

that each U_i is closed. It follows from the hypothesis of the theorem that X is normal. By Urysohn's extension theorem, U_i is C^* -embedded in X and consequently $\beta(U_i)$ is homeomorphic (and thus unimorphic) to $\text{cl}_{\beta X} U_i$. For this reason, we can write $\beta(U_i) = \text{cl}_{\beta X} U_i$. Now $\mathfrak{S} \mid (U_i \times V_j) \in [(\mu \times \mathfrak{F}_{\beta X}) \mid (U_i \times V_j)]_0^{(\alpha)}$ for all i and j . As $(\mu \times \mathfrak{F}_{\beta X})_0^{(\alpha)}$ is a quasiuniformity it is easy to see that for each i , the restriction of $\mathfrak{S}$ to $U_i \times \beta X$ belongs to $[(\mu \times \mathfrak{F}_{\beta X}) \mid (U_i \times \beta X)]_0^{(\alpha)}$; hence $\mathfrak{S} \mid (U_i \times \beta(U_i))$ belongs to $[(\mu \times \mathfrak{F}_{\beta X}) \mid (U_i \times \beta(U_i))]_0^{(\alpha)}$. Further, note that

$$G_1 \cap (U_i \times \beta(U_i)) = (U_i \times \beta(U_i)) \sim (X \times K) = (U_i \times \beta(U_i)) \sim (U_i \times K_i)$$

where $K_i = K \cap \beta(U_i)$ and

$$G_2 \cap (U_i \times \beta(U_i)) = (U_i \times \beta(U_i)) \sim \Delta(U_i).$$

Finally, note that $[(\mu \times \mathfrak{F}_{\beta X}) \mid (U_i \times \beta(U_i))]_0^{(\alpha)}$ is the same quasiuniformity as $(\mu \mid U_i \times \mathfrak{F}_{\beta(U_i)})_0^{(\alpha)}$. Therefore, the induction hypothesis applies and we can find a cover $\{W'_k\}_k$ in $(\mu \mid U_i)_0^{(\alpha)}$ such that $\text{cl}_{\beta(U_i)} W'_k \subset \beta(U_i) \sim K_i$ for all k . Now

$$\text{cl}_{\beta(U_i)} W'_k = \text{cl}_{\text{cl}_{\mu X} U_i} W'_k = \text{cl}_{\beta X} W'_k \cap \text{cl}_{\beta X} U_i.$$

Choose a closed cover $\{\tilde{W}'_l\}_l \in \mu_0^{(\alpha)}$ such that $\{U_i \cap \tilde{W}'_l\} < \{W'_k\}$. Then $\{U_i \cap \tilde{W}'_l\}_{l,l} \in \mu_0^{(\alpha)}/\mu = \mu_0^{(\alpha+1)}$ (since $\{U_i\} \in \mu$) and

$$\begin{aligned} \text{cl}_{\beta X}(U_i \cap \tilde{W}'_l) &= \text{cl}_{\beta X} U_i \cap \text{cl}_{\beta X} \tilde{W}'_l \subset \text{cl}_{\beta X} U_i \cap \text{cl}_{\beta X} W'_{k(l)} \\ &= \text{cl}_{\beta(U_i)} W'_{k(l)} \subset \beta(U_i) \sim K \subset \beta X \sim K \end{aligned}$$

since X is normal (see, for example, [6, p. 764]). Hence $P(\tau)$ is valid. This completes the induction. Since $\lambda(\mu \times \mathfrak{F}_{\beta X})$ is equinormal, we have $\mathfrak{S} \in \lambda(\mu \times \mathfrak{F}_{\beta X})$. As $P(\alpha)$ is valid for all α , it follows from Lemma 3 that there is a $\tilde{\mathfrak{S}} \in \lambda\mu$ such that $\text{cl}_{\beta X} V \cap K = \emptyset$ for all $V \in \tilde{\mathfrak{S}}$. By Lemma 2, μX is supercomplete.

Necessity follows directly from Exercise VII.8.c) in [5]: the product of a supercomplete space and a compact space is supercomplete. We give a proof for the reader's convenience. Let μX be supercomplete and let K be compact. Then X is paracompact and $\lambda\mu X = \mathfrak{F}X$. Hence $X \times K$ is paracompact. By Lemma 4, the inclusions $\mathfrak{F}X \times \mathfrak{F}K \subset \lambda\mu X \times \mathfrak{F}K \subset \lambda(\mu X \times \mathfrak{F}K) \subset \mathfrak{F}(X \times K)$ imply

$$\mathfrak{F}_{(X \times K)} = (\mathfrak{F}_X \times \mathfrak{F}_K)^{(1)} \subset (\lambda(\mu \times \mathfrak{F}_K))^{(1)} = \lambda(\mu \times \mathfrak{F}_K) \subset \mathfrak{F}_{(X \times K)}$$

and thus $\lambda(\mu \times \mathfrak{F}_K) = \mathfrak{F}_{(X \times K)}$. Hence, the proof is complete.

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