FIXED SUBALGEBRA OF A FROBENIUS ALGEBRA

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ABSTRACT. In this note, we study the fixed subalgebra of a Frobenius algebra under a finite group of automorphisms. If the automorphisms preserve the nondegenerate bilinear form, then the fixed subalgebra is also Frobenius.

1. Introduction. Let F be a fixed field of characteristic p including the case $p = 0$ and let A be a finite-dimensional Frobenius algebra over F . Let G be a finite group of algebraic automorphisms of A , i.e. automorphisms that fix the ground field, and let A^G be the fixed subalgebra under G . It has been shown [3] by J. L. Pascaud and J. Valette that it does not necessarily follow that A^G is Frobenius. In a recent paper [1], G. Azumaya has shown that if B is a finite-dimensional commutative algebra generated by a single element and if $A = B \otimes B$, then the fixed subalgebra of A under the involution $x \otimes y \rightarrow y \otimes x$ is Frobenius if and only if $p \neq 2$ or B is separable. On the other hand, by definition [2] a finite-dimensional algebra A with identity is Frobenius if and only if there is a nondegenerate associative bilinear form on A . The purpose of this note is to study the fixed subalgebra of an arbitrary Frobenius algebra under a finite group of algebraic automorphisms which preserves the bilinear form. Our main result is the following.

THEOREM 1.1. *Let A be a finite-dimensional Frobenius algebra with a nondegenerate associative bilinear form φ . Let G be a finite group of algebraic automorphisms of A such that $\varphi(gx, gy) = \varphi(x, y)$ for all $x, y \in A, g \in G$. Then the fixed subalgebra A^G is Frobenius provided $p \nmid |G|$.*

As an application, we will prove the following corollary which generalizes Azumaya's result.

COROLLARY 1.2. *Let B be a finite-dimensional Frobenius algebra over F and let $A = B \otimes B$. Let τ be the involution on A which is defined by $\tau(x \otimes y) = y \otimes x$. Then the fixed subalgebra A^G is also Frobenius provided $p \neq 2$.*

2. Bilinear forms of a vector space. In this section, let E be an n -dimensional vector space over F and let $\{l_1, \dots, l_n\}$ be a fixed basis for E . For a set of vectors $\{x_1, \dots, x_m\}$, we use $\langle x_1, \dots, x_m \rangle$ to denote the subspace generated by $\{x_1, \dots, x_m\}$. Since the proofs are easy, the following lemma and propositions are quoted without proofs.

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LEMMA 2.1. *Let φ be a bilinear form on E . Then φ is nondegenerate if and only if the $n \times n$ matrix $[\varphi(l_i, l_j)]$ is nonsingular.*

PROPOSITION 2.2. *Let φ be a bilinear form on E and let $\{x_1, \dots, x_m\}$ be a set of vectors. Then the restriction of φ to $\langle x_1, \dots, x_m \rangle$ is nondegenerate if and only if $\text{rank}[\varphi(x_i, x_j)] = \dim \langle x_1, \dots, x_m \rangle$.*

PROPOSITION 2.3. *For $i = 1, 2$, let φ_i be a nondegenerate bilinear form on vector spaces E_i . Then the tensor product $\varphi = \varphi_1 \otimes \varphi_2$ is also nondegenerate.*

3. A key proposition. As in §2, let E be an n -dimensional vector space over F . Let G be a finite group of automorphisms of E and let $E^G = \{x \in E \mid gx = x \text{ for all } g \in G\}$. We will always assume that $p \nmid |G|$. Again, we leave the proof of the following lemma to the readers.

LEMMA 3.1. *Let τ be an endomorphism of E which is defined by $\tau x = (1/|G|) \sum_{g \in G} gx$. Then $E^G = \text{Im } \tau$.*

Let φ be a bilinear form on E . Recall that an automorphism g of E is said to be an automorphism of φ if $\varphi(gx, gy) = \varphi(x, y)$ for all $x, y \in E$. The following proposition is the key in our study.

THEOREM 3.2. *Let φ be a nondegenerate bilinear form on E and let G be a finite group of automorphisms of φ . If $p \nmid |G|$ then the restriction of φ to E^G is also nondegenerate.*

PROOF. Let $\{l_1, \dots, l_n\}$ be a basis for E and let $\tau l_i = \sum_{k=1}^n a_{ik} l_k$. By Lemma 3.1, $\langle \tau l_1, \dots, \tau l_n \rangle = E^G$.

$$\begin{aligned} [\varphi(\tau l_i, \tau l_j)] &= \left[\varphi \left(\frac{1}{|G|} \sum_{g \in G} gl_i, \frac{1}{|G|} \sum_{h \in G} hl_j \right) \right] \\ &= \left[\frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in G} \varphi(gl_i, hl_j) \right] = \left[\frac{1}{|G|^2} \sum_{h \in G} \sum_{g \in G} \varphi(h^{-1}gl_i, l_j) \right] \\ &= \left[\frac{1}{|G|} \sum_{g \in G} \varphi(gl_i, l_j) \right] = [\varphi(\tau l_i, l_j)] = \left[\varphi \left(\sum_{k=1}^n a_{ik} l_k, l_j \right) \right] \\ &= \left[\sum_{k=1}^n a_{ik} \varphi(l_k, l_j) \right] = [a_{i,}] [\varphi(l_k, l_j)]. \end{aligned}$$

Note that $[\varphi(l_i, l_j)]$ is nonsingular since φ is nondegenerate. This implies that

$$\text{rank}[\varphi(\tau l_i, \tau l_j)] = \text{rank}[a_{i,}] = \dim \text{Im } \tau = \dim E^G.$$

It follows from Proposition 2.2 that $\varphi|_{E^G}$ is nondegenerate.

4. Proofs of Theorem 1.1 and Corollary 1.2. It is clear that Theorem 1.1 follows immediately from Theorem 3.2 and the definition of Frobenius algebra.

For Corollary 1.2 note that if ψ is a nondegenerate associative bilinear form on B , then $\varphi = \psi \otimes \psi$ is a nondegenerate associative bilinear form on $A = B \otimes B$. Then

$$\begin{aligned}\varphi(\tau(x \otimes y), \tau(x' \otimes y')) &= \varphi(y \otimes x, y' \otimes x') \\ &= \psi(y, y')\psi(x, x') = \varphi(x \otimes y, x' \otimes y').\end{aligned}$$

It follows that τ is also an automorphism of φ . It follows that the fixed algebra A^τ is Frobenius.

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