EFFECTIVE DETERMINATION OF THE DECOMPOSITION OF THE RATIONAL PRIMES IN A CUBIC FIELD

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ABSTRACT. The decomposition of the rational primes in a cubic field K is determined in terms of the coefficients of a defining polynomial of K . As a consequence, the discriminant D of K is straightforwardly computed and the cubic fields with index $i(K) = 2$ are easily characterized.

Introduction. The study of the decomposition of the rational primes in a cubic field K is well known if K is cyclic [3, §24] and the noncyclic case was developed by Hasse [5] and more recently by Martinet and Payan [6]. However, these results are not sufficient to determine such decompositions in terms of a defining polynomial of K . This is the problem we solve in this paper (Theorem 1). As a consequence, we obtain a straightforward computation of the discriminant of K (Theorem 2) and we reobtain in a completely elementary way a well-known theorem of Hasse [5, Theorem 6] on the possible values of the discriminant of a cubic field (Theorem 3). Finally, as another consequence of Theorem 1, we find a characterization of the cubic fields in which 2 is a common index divisor, which improves the given by Tornheim in [8, Theorem 2] (Theorem 4).

Most of our results are easily extendible to the case of relative cubic extensions, but we will not make these generalizations explicit.

Let K be a cubic field. We can suppose that $K = \mathbf{Q}(\theta)$, where θ is a root of an irreducible polynomial of the type

$$f(X) = X^3 - aX + b, \quad a, b \in \mathbf{Z}.$$

The discriminant of $f(X)$ is $\Delta = 4a^3 - 27b^2$ and if we denote by D the discriminant of K we have $\Delta = i(\theta)^2 \cdot D$, where $i(\theta)$ denotes the index of θ .

For every prime $p \in \mathbf{Z}$ and integer $m \in \mathbf{Z}$ we denote by $v_p(m)$ the greatest exponent k such that $p^k \mid m$. If for any prime p we have

$$(1) \quad v_p(a) \geq 2 \quad \text{and} \quad v_p(b) \geq 3,$$

then θ/p is an algebraic integer whose equation is $X^3 - (a/p^2)X + (b/p^3)$. Therefore, we can assume that (1) is not satisfied for any prime p .

Let $s_p = v_p(\Delta)$ and $\Delta_p = \Delta/p^{s_p}$ for every prime p .

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THEOREM 1. *The primes of $\mathbf{Q}$ decompose in K as follows:
Decomposition of 2.*

$$\begin{array}{ll}
 a, b \text{ even:} & \begin{cases} 1 \leq v_2(b) \leq v_2(a) & \dots & 2 = P^3. \\ 1 = v_2(a) < v_2(b) & \dots & 2 = P \cdot Q^2. \end{cases} \\
 a \text{ even, } b \text{ odd:} & \dots \dots \dots 2 = P \cdot Q. \\
 a \text{ odd, } b \text{ even:} & \begin{cases} s_2 \text{ odd:} & \dots & 2 = P \cdot Q^2. \\ s_2 \text{ even:} & \begin{cases} \Delta_2 \equiv 3 \pmod{4} & \dots & 2 = P \cdot Q^2. \\ \Delta_2 \equiv 5 \pmod{8} & \dots & 2 = P \cdot Q. \\ \Delta_2 \equiv 1 \pmod{8} & \dots & 2 = P \cdot Q \cdot R. \end{cases} \end{cases} \\
 a, b \text{ odd:} & \dots \dots \dots 2 = P.
 \end{array}$$

Decomposition of 3.

$$\begin{array}{ll}
 3 \mid a, 3 \mid b: & \begin{cases} 1 \leq v_3(b) \leq v_3(a) & \dots & 3 = P^3. \\ 1 = v_3(a) < v_3(b) & \dots & 3 = P \cdot Q^2. \end{cases} \\
 3 \nmid a: & \begin{cases} a \equiv -1 \pmod{3} & \dots & 3 = P \cdot Q. \\ a \equiv 1 \pmod{3} & \begin{cases} 3 \nmid b & \dots & 3 = P. \\ 3 \mid b & \dots & 3 = P \cdot Q \cdot R. \end{cases} \end{cases} \\
 3 \mid a, 3 \nmid b: & \begin{cases} a \not\equiv 3 \pmod{9} & \begin{cases} b^2 \equiv a+1 \pmod{9} & \dots & 3 = P \cdot Q^2. \\ b^2 \not\equiv a+1 \pmod{9} & \dots & 3 = P^3. \end{cases} \\ a \equiv 3 \pmod{9} & \begin{cases} b^2 \equiv a+1 \pmod{27} & \begin{cases} s_3 \text{ odd:} & \dots & 3 = P \cdot Q^2. \\ s_3 \text{ even:} & \begin{cases} \Delta_3 \equiv -1 \pmod{3} & \dots & 3 = P \cdot Q. \\ \Delta_3 \equiv 1 \pmod{3} & \begin{cases} s_3 = 6 & 3 = P. \\ s_3 > 6 & 3 = P \cdot Q \cdot R. \end{cases} \end{cases} \end{cases} \\ b^2 \not\equiv a+1 \pmod{27} & \dots & 3 = P^3. \end{cases} \end{cases}
 \end{array}$$

Decomposition of $p > 3$.

$$\begin{array}{ll}
 p \mid a, p \mid b: & \begin{cases} 1 \leq v_p(b) \leq v_p(a) & \dots & p = P^3. \\ 1 = v_p(a) < v_p(b) & \dots & p = P \cdot Q^2. \end{cases} \\
 p \mid a, p \nmid b: & \begin{cases} p \equiv -1 \pmod{3} & \dots & p = P \cdot Q. \\ p \equiv 1 \pmod{3} & \begin{cases} (b/p)_3 = 1 & \dots & p = P \cdot Q \cdot R. \\ (b/p)_3 \neq 1 & \dots & p = P. \end{cases} \end{cases} \\
 p \nmid a, p \mid b: & \begin{cases} (a/p) = 1 & \dots & p = P \cdot Q \cdot R. \\ (a/p) = -1 & \dots & p = P \cdot Q. \end{cases} \\
 p \nmid ab: & \begin{cases} s_p \text{ odd:} & \dots & p = P \cdot Q^2. \\ s_p \text{ even:} & \begin{cases} (\Delta_p/p) = 1 & \begin{cases} f(X) \text{ has some root (mod. } p) & \dots & p = P \cdot Q \cdot R. \\ f(X) \text{ has no roots (mod. } p) & \dots & p = P. \end{cases} \\ (\Delta_p/p) = -1 & \dots & p = P \cdot Q. \end{cases} \end{cases}
 \end{array}$$

REMARK. This theorem is particularly interesting for $p = 2$ and 3 ; not only because the exceptional behaviour of these primes makes its study more difficult, but specially because it solves these cases in a completely effective way. For $p > 3$ effectiveness depends on checking if an integer is a quadratic or cubic residue (mod. p) and on checking if the polynomial has roots (mod. p). Nevertheless, we note that for small primes effectiveness is also complete since the quadratic and cubic residues can be easily computed and because we have found an interesting method, to be developed in a later paper, to determine whether $f(X)$ has roots (mod. p) or not. Thus, if $p \nmid ab$, s_p even and $(\Delta_p/p) = 1$ we have found, for instance, that

$$\begin{aligned} 5 &= P \cdot Q \cdot R. \Leftrightarrow a^3 \equiv 3b^2 \pmod{5}, \\ 7 &= P \cdot Q \cdot R. \Leftrightarrow a^3 \equiv 5b^2 \pmod{7}, \\ 11 &= P \cdot Q \cdot R. \Leftrightarrow a^3 \equiv 4b^2 \text{ or } 8b^2 \pmod{11}, \\ 13 &= P \cdot Q \cdot R. \Leftrightarrow a^3 \equiv 7b^2 \text{ or } 10b^2 \pmod{13}. \end{aligned}$$

For the proof of Theorem 1 we need the following lemmas.

LEMMA 1 (DEDEKIND [2, §2]). Let $g(X) \in \mathbf{Z}[X]$ be an irreducible polynomial, ω a root of $g(X)$ and $L = \mathbf{Q}(\omega)$. Let $p \in \mathbf{Z}$ be a prime and let

$$g(X) \equiv \varphi_1(X)^{e_1} \cdots \varphi_r(X)^{e_r} \pmod{p}$$

be the factorization of $g(X)$ into irreducible factors (mod. p). If $p \nmid i(\omega)$, then $p = P_1^{e_1} \cdots P_r^{e_r}$ in L ; and for all i , $f(P_i/p) = \deg(\varphi_i(X))$ and $P_i = (p, \varphi_i(\omega))$.

LEMMA 2 (DEDEKIND [2, §3]). With the same notations of Lemma 1, let

$$g(X) = \varphi_1(X)^{e_1} \cdots \varphi_r(X)^{e_r} + p \cdot M(X), \quad M(X) \in \mathbf{Z}[X].$$

Then, $p \nmid i(\omega)$ is equivalent to $\varphi_i(X) \nmid M(X) \pmod{p}$, for every i such that $e_i > 1$.

LEMMA 3 (BAUER [1]). Let $g(X)$, ω , L , p , be as in Lemma 1. For every prime P of L lying over p , the quotient $v_P(\omega)/e(P/p)$ is equal to the slope of one of the sides of Newton's polygon of $g(X)$ with respect to p and conversely, if $\lambda \in \mathbf{Q}$ is the slope of one side of that polygon, there exists a prime P of L lying over p such that $v_P(\omega)/e(P/p) = \lambda$.

The following lemma is an obvious consequence, in the cubic case, of more general results of Dedekind and Hensel (see [7, Theorem 4.8 and Proposition 6.10]).

LEMMA 4. With the same notations of Theorem 1:

If $p = P \cdot Q^2$ in K and $p \neq 2$, then $v_p(D) = 1$.

If $p = P^3$ in K and $p \neq 3$, then $v_p(D) = 2$.

If $2 = P \cdot Q^2$ in K , then $v_2(D) = 2$ or 3 .

If $3 = P^3$ in K , then $v_3(D) = 3, 4$ or 5 .

We also need the following easy result.

LEMMA 5. *With the same notations of Theorem 1:*

If $p = P$ or $p = P \cdot Q \cdot R$ in K , then $\Delta \in \mathbb{Q}_p^2$.

If $p = P \cdot Q$ or $p = P \cdot Q^2$ in K , then $\Delta \notin \mathbb{Q}_p^2$.

PROOF OF THEOREM 1.

Case I. $p \mid a, p \nmid b$. By hypothesis $2v_p(b) \neq 3v_p(a)$, hence, the only possibilities are $0 < 2v_p(b) < 3v_p(a)$ and $0 < 3v_p(a) < 2v_p(b)$. In the first case, $v_p(b) = 1$ or 2 and $v_p(a) \geq v_p(b)$, so that Newton's polygon of $f(X)$ with respect to p has only one side with slope $\frac{1}{3}$ or $\frac{2}{3}$, hence, by Lemma 3, $p = P^3$. In the second case, $1 = v_p(a) < v_p(b)$ and the polygon has two sides, one of them with slope $\frac{1}{2}$; again by Lemma 3, we have $p = P \cdot Q^2$.

Case II. $p \nmid a, p \mid b$. In this case, by Lemma 3 there are at least two primes of K over p . If $p \neq 2$, then $s_p = 0$ and by Lemma 1, $p = P \cdot Q \cdot R$ or $p = P \cdot Q$ according to $(a/p) = 1$ or -1 .

If $p = 2$, then $s_2 > 0$. If s_2 is odd, then 2 is ramified, so that $2 = P \cdot Q^2$. Let us assume that s_2 is even. Now, by Lemma 5, $2 = P \cdot Q \cdot R$ if and only if Δ is a square in $\mathbb{Q}_2$, that is, if and only if $\Delta_2 \equiv 1 \pmod{8}$. Finally, if $\Delta_2 \not\equiv 1 \pmod{8}$, we have $f(X) = g(X) \cdot h(X)$ in $\mathbb{Q}_2[X]$, with $\deg(g(X)) = 1$, $\deg(h(X)) = 2$, both irreducible in $\mathbb{Q}_2[X]$. We have $2 = P \cdot Q$ if and only if the quadratic extension of $\mathbb{Q}_2$ which define the roots of $h(X)$ is nonramified and that happens if and only if $\Delta_2 \equiv 5 \pmod{8}$ since $\mathbb{Q}_2(\sqrt{5})$ is the only nonramified quadratic extension of $\mathbb{Q}_2$ [7, p. 215].

Case III. $p \mid a, p \nmid b$. If $p \neq 3$, then $s_p = 0$. Hence, by Lemma 1 the decomposition of p depends only on the factorization of $f(X) \equiv X^3 + b \pmod{p}$. If $p \equiv -1 \pmod{3}$, $f(X)$ has only one root $\pmod{p}$, hence $p = P \cdot Q$. If $p \equiv 1 \pmod{3}$, $f(X)$ has three roots $\pmod{p}$ or none according to b being a cubic residue $\pmod{p}$ or not.

For $p = 3$, $f(X) \equiv (X + b)^3 \pmod{3}$. Making the substitution $\theta_1 = \theta + b$ we obtain the equation

$$f_1(X) = X^3 - 3bX^2 + (3b^2 - a)X - (b^3 - ab - b).$$

If $a \not\equiv 3 \pmod{9}$, we have $3b^2 \equiv 3 \not\equiv a \pmod{9}$ and Newton's polygon of $f_1(X)$ with respect to 3 gives a complete answer to our problem. In fact, if $b^2 \equiv a + 1 \pmod{9}$, then $3 = P \cdot Q^2$ since the polygon has two sides, one of them with slope $\frac{1}{2}$; and if $b^2 \not\equiv a + 1 \pmod{9}$, then $3 = P^3$ since the polygon has only one side with slope $\frac{1}{3}$.

Assume now that $a \equiv 3 \pmod{9}$. If $b^2 \not\equiv a + 1 \pmod{27}$, then $3 = P^3$ since the polygon has only one side with slope $\frac{1}{3}$ or $\frac{2}{3}$. If $b^2 \equiv a + 1 \pmod{27}$, making the substitution $\theta_2 = \theta_1/3$ we obtain the equation

$$f_2(X) = X^3 - bX^2 + \frac{3b^2 - a}{9}X - \frac{b^3 - ab - b}{27}.$$

Observe first that the congruence $f_2(X) \equiv (X + c)^3 \pmod{3}$, $c \in \mathbb{Z}$, is not possible, hence, $3 \neq P^3$, and $3 = P$ if and only if $f_2(X)$ is irreducible $\pmod{3}$. Since $3 \neq P^3$, by Lemma 4 we have $3 = P \cdot Q^2$ if and only if s_3 is odd; and by Lemma 5, $3 = P \cdot Q$ if and only if s_3 is even and $\Delta_3 \equiv -1 \pmod{3}$.

Finally, suppose that s_3 is even and $\Delta_3 \equiv 1 \pmod{3}$ and let $D_2 = \Delta/27^2$ be the discriminant of $f_2(X)$. Under our assumptions $f_2(X)$ cannot have only one root $\pmod{3}$ and since $b \not\equiv 0 \pmod{3}$ it cannot have three simple roots $\pmod{3}$, hence,

$$s_3 = 6 \Leftrightarrow 3 \nmid D_2 \Leftrightarrow f_2(X) \text{ irreducible } \pmod{3} \Leftrightarrow 3 = P.$$

Case IV. $p \nmid ab$. If $p = 2$ or 3 , then $s_p = 0$ and the assertions of Theorem 1 are an obvious consequence of Lemma 1. Let $p > 3$. By Lemma 4 we have $p = P \cdot Q^2$ if and only if s_p is odd. Moreover, since the congruence

$$f(X) \equiv (X + c)^3 \pmod{p}, \quad c \in \mathbb{Z},$$

is not possible, we can assure that $p \neq P^3$. Hence, by Lemma 4 we have $p = P \cdot Q$ if and only if s_p is even and $(\Delta_p/p) = -1$. If s_p is even and $(\Delta_p/p) = 1$, then $p = P$ or $P \cdot Q \cdot R$ according to $f(X)$ being irreducible $\pmod{p}$ or not.

This ends the proof of Theorem 1.

We now apply Theorem 1 to obtain a direct computation of D in terms of a and b .

THEOREM 2. *With the same notations, for every prime $p > 3$ we have*

$$\begin{aligned} v_p(D) = 2 & \Leftrightarrow 1 \leq v_p(b) \leq v_p(a), \\ v_p(D) = 1 & \Leftrightarrow s_p \text{ odd}, \\ v_p(D) = 0 & \text{ otherwise;} \end{aligned}$$

for $p = 2$ we have

$$\begin{aligned} v_2(D) = 3 & \Leftrightarrow s_2 \text{ odd}, \\ v_2(D) = 2 & \Leftrightarrow \begin{cases} 1 \leq v_2(b) \leq v_2(a), \text{ or} \\ s_2 \text{ even and } \Delta_2 \equiv 3 \pmod{4}, \end{cases} \\ v_2(D) = 0 & \text{ otherwise;} \end{aligned}$$

and for $p = 3$ we have

$$\begin{aligned} v_3(D) = 5 & \Leftrightarrow 1 \leq v_3(b) < v_3(a), \\ v_3(D) = 4 & \Leftrightarrow \begin{cases} v_3(a) = v_3(b) = 2, \text{ or} \\ a \equiv 3 \pmod{9}, 3 \nmid b \text{ and } b^2 \not\equiv 4 \pmod{9}, \end{cases} \\ v_3(D) = 3 & \Leftrightarrow \begin{cases} v_3(a) = v_3(b) = 1, \text{ or} \\ 3 \mid a, 3 \nmid b, a \not\equiv 3 \pmod{9} \text{ and } b^2 \not\equiv a + 1 \pmod{9}, \text{ or} \\ a \equiv 3 \pmod{9}, b^2 \equiv 4 \pmod{9} \text{ and } b^2 \not\equiv a + 1 \pmod{27}, \end{cases} \\ v_3(D) = 1 & \Leftrightarrow \begin{cases} 1 = v_3(a) < v_3(b), \text{ or} \\ 3 \mid a, a \not\equiv 3 \pmod{9} \text{ and } b^2 \equiv a + 1 \pmod{9}, \text{ or} \\ a \equiv 3 \pmod{9}, b^2 \equiv a + 1 \pmod{27} \text{ and } s_3 \text{ odd}, \end{cases} \\ v_3(D) = 0 & \Leftrightarrow \begin{cases} 3 \nmid a, \text{ or} \\ a \equiv 3 \pmod{9}, b^2 \equiv a + 1 \pmod{27} \text{ and } s_3 \text{ even}. \end{cases} \end{aligned}$$

PROOF. We prove only the results concerning $v_3(D) \geq 3$, which correspond to $3 = P^3$, since all the other cases are an easy consequence of Theorem 1 and Lemma 4.

Since $v_3(D) = 3$ if $v_3(\Delta) = 3$ and $v_3(D) = 4$ if $v_3(\Delta)$ is even, the computation of $v_3(\Delta)$ solves all the cases of $3 = P^3$ except for the following:

- (i) $1 = v_3(b) < v_3(a)$,
- (ii) $2 = v_3(b) < v_3(a)$,
- (iii) $a \equiv 3 \pmod{9}$, $b^2 \equiv 4 \pmod{9}$ and $b^2 \not\equiv a + 1 \pmod{27}$.

In case (i), $v_3(\Delta) = 5$ and by Lemma 2 we have $v_3(D) = 5$ because $3 \nmid i(\theta)$. In case (ii) we have $v_3(\Delta) = 7$. The element $\theta' = \theta^2/3$ is an integer and its irreducible polynomial is $X^3 - (2a/3)X^2 + (a^2/9)X - (b^2/27)$. By Lemma 2 we have $3 \nmid i(\theta')$ and it is easy to see that $i(\theta) = 3 \cdot i(\theta')$, hence $v_3(D) = 5$. In case (iii) we have $v_3(\Delta) = 5$ and by Lemma 2, $3 \mid i(\theta)$, so that $v_3(D) = 3$.

REMARK. Clearly, this theorem enables us to compute at once $i(\theta) = \sqrt{(\Delta/D)}$ in terms of a and b . That shortens considerably the computation of a basis of the integers of K following the method of Voronoi (see [3, §17]).

Theorem 2 provides also an elementary proof of the following theorem, which was proved first by Hasse [5] and later by Martinet and Payan [6] using more elaborate arguments.

THEOREM 3 (HASSE). If d denotes the discriminant of $\mathbf{Q}(\sqrt{D})$ we have

$$(2) \quad D = d \cdot 3^{2m} \cdot f^2,$$

where,

- (i) $f = p_1 \cdots p_g$ is a square-free integer, $(f, 3d) = 1$ and $p_i \equiv (d/p_i) \pmod{3}$ for all i .
- (ii) $0 \leq m \leq 2$; if $3 \nmid d$, then $m \neq 1$ and if $d \equiv 3 \pmod{9}$, then $m \neq 2$.

PROOF. Let $D = d_0 \cdot 3^{2m} \cdot f_0^2$, d_0 square-free and $3 \nmid f_0$. If $d_0 \equiv 1 \pmod{4}$, we have $d = d_0$. If $d_0 \not\equiv 1 \pmod{4}$, then $D \not\equiv 1 \pmod{4}$, hence, by Stickelberger's criterion (see [7, Theorem 2.5]) we have $D \equiv 0 \pmod{4}$, so that $2 \mid f_0$ and $d = 4d_0$ divides D . In all cases we can write D as in (2).

By Lemma 4 f is square-free and $(f, 3d) = 1$. Let $p \neq 3$; we find

$$p \mid f \Leftrightarrow \begin{cases} v_p(D) = 2, & \text{if } p > 3 \\ v_2(D) = 2 & \text{and } 2 \nmid d, & \text{if } p = 2 \end{cases}$$

$$\stackrel{(\text{Theorem 2})}{\Leftrightarrow} 1 \leq v_p(b) \leq v_p(a) \stackrel{(\text{Theorem 1})}{\Leftrightarrow} p = P^3.$$

Therefore, if $p \mid f$ and $p > 3$, we have $(d/p) = (\Delta_p/p) = (-3/p) \equiv p \pmod{3}$. And if $2 \mid f$, then $d \equiv \Delta_2 \equiv 5 \pmod{8}$, so that $(d/2) = -1 \equiv 2 \pmod{3}$. That ends the proof of (i).

By Lemma 4, $0 \leq m \leq 2$ and $m \neq 1$ if $3 \nmid d$. If $3 \mid d$, then $(d/3) \equiv \Delta_3 \pmod{3}$. Now, if $3 \mid d$ and $m = 2$, by Theorem 2 we have $1 \leq v_3(b) < v_3(a)$ and in this case, always $\Delta_3 \equiv -1 \pmod{3}$, so that $d \equiv 6 \pmod{9}$. That ends the proof of (ii).

Finally, we find a characterization of the cubic fields in which 2 is a common index divisor. We recall that the index $i(L)$ of any number field L is defined as

$$i(L) = \text{g.c.d.}\{i(\omega) \mid \omega \in L \text{ is an integer such that } L = \mathbf{Q}(\omega)\}.$$

It is well known that if K is a cubic field, then $i(K) = 1$ or 2 and that $i(K) = 2$ if and only if $2 = P \cdot Q \cdot R$ in K [4]. Therefore, our Theorem 1 furnishes the following simple criterion.

THEOREM 4. *With the same notations of Theorem 1,*

$$i(K) = 2 \Leftrightarrow a \text{ odd, } b \text{ even, } s_2 \text{ even and } \Delta_2 \equiv 1 \pmod{8}.$$

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