NUMBERS OF SOLUTIONS OF CONGRUENCES: POINCARÉ SERIES FOR STRONGLY NONDEGENERATE FORMS

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ABSTRACT. Let p be a fixed prime, $f(x_1, \dots, x_k)$ a polynomial over $\mathbf{Z}_p$, the p -adic integers, c_n the number of solutions of $f = 0$ over $\mathbf{Z}/p^n\mathbf{Z}$ and $P_f(t) = \sum_{i=0}^{\infty} c_i t^i$ the Poincaré series. A general approach to computing c_n and $P_f(t)$ is given and explicit formulas are derived for strongly nondegenerate forms.

1. Introduction. Let p be a fixed prime and $f(x) = f(x_1, \dots, x_k)$ a polynomial with coefficients in $\mathbf{Z}_p$, the p -adic integers. Let c_n denote the number of solutions of $f = 0$ over the ring $\mathbf{Z}/p^n\mathbf{Z}$, with $c_0 = 1$. Then the *Poincaré series* $P_f(t)$ is the generating function

$$P_f(t) = \sum_{i=0}^{\infty} c_i t^i.$$

This series was introduced by Borevich and Shafarevich [1, p. 47] and is the analogue for the rings $\mathbf{Z}/p^n\mathbf{Z}$ of the zeta function of f over finite fields.

Borevich and Shafarevich conjectured that $P_f(t)$ is a rational function of t for all polynomials. This was proved by Igusa in 1975, in a more general setting, using a mixture of analytic and algebraic methods in which a key tool is the resolution of singularities of algebraic varieties [2, 3]. The proof is nonconstructive and further studies of the problem by Igusa are summarized in his Tata Notes [4]. D. Meuser [6] extended Igusa's theorem from hypersurfaces to algebraic varieties. E. Stevenson [7] treated polynomials of the form $\sum x_i^{d_i}$ where $p \nmid d_i$, using Jacobi sums. Her methods, which are fully constructive, are quite complicated for carrying out computations.

In this paper an elementary algebraic-combinatorial approach is given for deriving a recursion for c_n and computing the Poincaré series for strongly nondegenerate forms. In future papers I will present an analogous approach to polynomials in one variable and classes of algebraic curves. In §2 some general remarks and a lemma which is fundamental for counting solutions are presented. In §3 the results are derived for strongly nondegenerate forms.

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2. Fundamental Lemma. A solution of $f(x) = 0$ over $\mathbf{Z}/p^n\mathbf{Z}$ can be represented as a k -tuple $x = (x_1, \dots, x_k)$, $0 \leq x_i \leq p^n - 1$, where we also think of each x_i with its unique expansion in base p :

$$x_i = \sum_{j=0}^{n-1} a_j^{(i)} p^j, \quad 0 \leq a_j^{(i)} \leq p-1.$$

Since $f(x) \equiv 0 \pmod{p^n}$ implies that $f(x) \equiv 0 \pmod{p^l}$ for all $l < n$ we also have that every such x is congruent to a unique x' in $(\mathbf{Z}/p^l\mathbf{Z})^k \pmod{p^l}$. In a base p expansion each entry in x' consists of the first $l-1$ terms of the corresponding entry in x . We say that x is a *descendant* of x' .

The fact that each solution $(x_1, \dots, x_k) \pmod{p^n}$ can branch to a finite number of descendants $\pmod{p^{n+1}}$, all of the form $(\dots, x_i + b_i p^n, \dots)$, is the basis of our counting arguments.

DEFINITION. A solution $a = (a_1, \dots, a_k)$ in $(\mathbf{Z}/p\mathbf{Z})^k$ of $f \equiv 0 \pmod{p}$ is *nonsingular* if the partial derivatives $f_{x_i}(a)$ are not all $\equiv 0 \pmod{p}$; otherwise the solution is *singular*.

A basic step in our counting arguments is provided by the following simple application of Taylor's theorem.

FUNDAMENTAL LEMMA. Let $f(x_1, \dots, x_k)$ be a polynomial over $\mathbf{Z}_p$ and let $a = (a_1, \dots, a_k)$ in $(\mathbf{Z}/p\mathbf{Z})^k$ be a nonsingular solution of $f \equiv 0 \pmod{p}$. Then there are $p^{(n-1)(k-1)}$ solutions of

$$(1) \quad f \equiv 0 \pmod{p^n},$$

i.e. solutions in $(\mathbf{Z}/p^n\mathbf{Z})^k$, descending from a .

PROOF. Let $a^{(n)} = (a_1^{(n)}, \dots, a_k^{(n)})$ be a solution of (1) descending from $a = a^{(1)}$. We shall prove that there are p^{k-1} solutions $a^{(n+1)}$, of $f \equiv 0 \pmod{p^{n+1}}$, descending from $a^{(n)}$. The lemma then follows by a simple induction.

Thinking of our numbers as expanded in the base p we must decide for which b_i , $0 \leq b_i \leq p-1$, the vector $a^{(n+1)} = (a_1^{(n)} + b_1 p^n, \dots, a_k^{(n)} + b_k p^n)$ will be a solution of $f \equiv 0 \pmod{p^{n+1}}$. By Taylor's theorem

$$(2) \quad f(a^{(n)} + bp^n) = f(a^{(n)}) + \sum_i f_{x_i}(a^{(n)}) b_i p^n + \text{higher order terms}.$$

Now $f(a^{(n)}) \equiv 0 \pmod{p^n}$ and the higher order terms contain p to a power larger than n . Therefore, recalling that $a_i^{(n)} \equiv a_i^{(1)} \pmod{p}$ and dividing both sides of (2) by p^n , we have

$$f(a^{(n+1)}) \equiv 0 \pmod{p^{n+1}},$$

if and only if

$$c + \sum_i f_{x_i}(a^{(1)}) b_i \equiv 0 \pmod{p}.$$

But by assumption $f_{x_i}(a^{(1)}) \not\equiv 0$ for some i , say $i = l$. Therefore for any choice of the b_i , $i \neq l$, we can solve for $b_l \pmod{p}$. Since there are p choices for each b_i , there are p^{k-1} solutions of $f \equiv 0 \pmod{p^{n+1}}$ descending from $a^{(n)}$. Q.E.D.

Since a solution of $f \equiv 0 \pmod{p^n}$ "branches" to its descendants mod p^{n+1} it can be very useful to interpret our problems using the combinatorial structure of rooted trees. If a is a solution of $f = 0$ in $(\mathbf{Z}/p\mathbf{Z})^k$, then we let a be the rooted vertex of a tree and take all its descendants as the other vertices. Every solution mod p^n is connected by edges oriented outward to each of its descendants mod p^{n+1} .

PROPOSITION. *If a is a solution of $f \equiv 0 \pmod{p}$ which has infinitely many descendants then there is a solution $\bar{a}$ in $\mathbf{Z}_p^k$ of $f = 0$ such that $\bar{a} \equiv a \pmod{p}$.*

PROOF. The tree T associated to a , as described above, is an infinite tree by assumption. However each vertex of T is incident to only a finite number of vertices since there are only a finite number of solutions of $f \equiv 0 \pmod{p^n}$ for each n . Therefore, by König's lemma [5, p. 38] there exists an infinite directed branch beginning at a with vertices $a = a^{(1)}, a^{(2)}, \dots$, which satisfy $f(a^{(n)}) \equiv 0 \pmod{p^n}$ and $a^{(n+1)} \equiv a^{(n)} \pmod{p^n}$. Hence there exist integers $\{c_i^{(j)}, j = 1, \dots, k, i = 0, 1, \dots\}$, with $0 \leq c_i^{(j)} \leq p - 1$, such that

$$a^{(n)} = \left(\sum_{i=0}^{n-1} c_i^{(1)} p^i, \dots, \sum_{i=0}^{n-1} c_i^{(k)} p^i \right).$$

Let $\bar{a} = (\sum_{i=0}^{\infty} c_i^{(1)} p^i, \dots, \sum_{i=0}^{\infty} c_i^{(k)} p^i)$. Then $\bar{a}$ is in $\mathbf{Z}_p^k$ and by the continuity of f , $f(\bar{a}) = 0$. Clearly $\bar{a} \equiv a \pmod{p}$.

A well-known result [1, p. 43] which follows immediately is

COROLLARY. *If a is a nonsingular solution of $f \equiv 0 \pmod{p}$, then there exists an $\bar{a}$ in $\mathbf{Z}_p^k$ such that $f(\bar{a}) = 0$ and $\bar{a} \equiv a \pmod{p}$.*

PROOF. By the fundamental lemma a has infinitely many descendants and the Proposition applies.

3. Strongly nondegenerate forms. By a *strongly nondegenerate* form we mean a homogeneous polynomial $F(x_1, \dots, x_k)$ such that the only singular solution of $F \equiv 0 \pmod{p}$ is $(0, 0, \dots, 0)$. Examples of such forms include $\sum_{i=1}^k e_i x_i^d$, where $p \nmid d$ and the e_i are p -adic units, and $x^2 + y^2 + xy$ where $p \neq 2, 3$.

THEOREM. *Let $F(x_1, \dots, x_k)$ be a strongly nondegenerate form of degree d with coefficients in $\mathbf{Z}_p$. Let c_n be the number of solutions of $F = 0$ in $(\mathbf{Z}/p^n\mathbf{Z})^k$, with $c_0 = 1$. Then*

(A) *we have*

$$(3) \quad c_n = (c_1 - 1)p^{(n-1)(k-1)} + p^{k(n-1)}$$

for $1 \leq n \leq d$,

$$(4) \quad c_n = (c_1 - 1)p^{(n-1)(k-1)} + p^{k(d-1)}c_{n-d}$$

for $d < n$.

(B) *The Poincaré series*

$$P_F(t) = \frac{R(t)}{(1 - p^{k-1}t)(1 - p^{k(d-1)}t^d)}$$

where $R(t)$ is a polynomial of degree d , which is effectively, and easily, computable.

Igusa [4, p. 104] gave a different derivation for the denominator of $P_F(t)$.

PROOF. To derive the recursions in (A) let $c_n = N_n + S_n$ where N_n denotes the number of solutions mod p^n descending from nonsingular solutions mod p and S_n denotes the number of solutions mod p^n descending from $(0, \dots, 0)$, the only singular solution mod p .

Since $(0, \dots, 0)$ is the only singular solution mod p there are $c_1 - 1$ nonsingular solutions mod p and by the fundamental lemma

$$N_n = (c_1 - 1)p^{(n-1)(k-1)}.$$

To compute S_n , we work by cases.

(i) $n = 1$.

By definition $S_n = 1$.

(ii) $2 \leq n \leq d$.

Let

$$a = \left(\sum_{i=1}^{n-1} a_i^{(1)} p^i, \dots, \sum_{i=1}^{n-1} a_i^{(k)} p^i \right),$$

$0 \leq a_i^{(j)} \leq p - 1$, be an arbitrary vector $\equiv 0 \pmod{p}$. Then $a = pa^*$ where $a^* = (\sum_{i=1}^{n-1} a_i^{(1)} p^{i-1}, \dots, \sum_{i=1}^{n-1} a_i^{(k)} p^{i-1})$. Since $d \geq n$, we have, by the homogeneity of F ,

$$F(a) = F(pa^*) = p^d F(a^*) \equiv 0 \pmod{p^n}.$$

Thus all such vectors are solutions and since there are $k(n-1)$ coefficients $a_i^{(j)}$ and p choices for each coefficient we have $S_n = p^{k(n-1)}$.

(iii) $d < n$.

Using the notation of case (ii)

$$F(a) = p^d F(a^*) \equiv 0 \pmod{p^n}$$

if and only if $F(a^*) \equiv 0 \pmod{p^{n-d}}$.

But $p^{n-d}, p^{n-d+1}, \dots, p^{n-2}$ are $\equiv 0 \pmod{p^{n-d}}$ and we have free choice for each of their $d-1$ coefficients in each term of a^* yielding $p^{k(d-1)}$ choices. We want the remaining part of a^* , viz. $(\sum_{i=1}^{n-d-1} a_i^{(1)} p^{i-1}, \dots, \sum_{i=1}^{n-d-1} a_i^{(k)} p^{i-1})$ to be an arbitrary solution of $F \equiv 0 \pmod{p^{n-d}}$ and there are exactly c_{n-d} such solutions.

Hence $S_n = p^{k(d-1)} c_{n-d}$ and part (A) is proved.

To compute $P_F(t)$ we form a generating function using equations (3) and (4). Note that, since $c_0 = 1$, (3) and (4) agree for $n = d$. Thus we have

$$\sum_{n=d}^{\infty} c_n t^n = (c_1 - 1)t \sum_{n=d}^{\infty} p^{(n-1)(k-1)} t^{n-1} + p^{k(d-1)} t^d \sum_{n=d}^{\infty} c_{n-d} t^{n-d}$$

or

$$P_F(t) - \sum_{n=0}^{d-1} c_n t^n = \frac{(c_1 - 1)t^n (p^{(k-1)} t)^{d-1}}{(1 - p^{k-1} t)} + p^{k(d-1)} t^d P_F(t).$$

Thus letting $Q_F(t) = \sum_{n=0}^{d-1} c_n t^n$ we have

$$P_F(t) = \frac{(c_1 - 1)p^{(k-1)(d-1)} t^d + (1 - p^{k-1} t) Q_F(t)}{(1 - p^{k-1} t)(1 - p^{k(d-1)} t^d)}$$

where $Q_F(t)$ is a polynomial of degree $d - 1$ which can be computed directly from equation (3) by summing geometric series. By looking at the expression for c_{d-1} in (3) we see that the numerator of $P_F(t)$ is a polynomial of degree d .

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