A SIMPLE CONSTRUCTION FOR RIGID AND WEAKLY HOMOGENEOUS BOOLEAN ALGEBRAS ANSWERING A QUESTION OF RUBIN^{1,2}

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ABSTRACT. We introduce a method for constructing Boolean algebras from trees which preserves some of the trees' properties. The method is used to produce a very simple construction for rigid Boolean algebras and to construct a weakly homogeneous Boolean algebra without homogeneous factors.

0. Introduction. In this paper we present two constructions: a weakly homogeneous Boolean algebra that answers a question of Rubin, and what we feel is the simplest construction of a rigid Boolean algebra.

A rigid Boolean algebra is one which has no nonidentity automorphism. We begin our construction of a rigid Boolean algebra by taking a tree T satisfying (a) the height of T is ω ; (b) T has a single root; (c) for all distinct $\alpha, \beta \in T$, the number of immediate successors to α is different from the number of immediate successors to β ; and (d) the number of immediate successors is always regular. We form "wedges", $S_\alpha = \{\beta \in T: \alpha \leq \beta\}$, for each $\alpha \in T$, and close the set of wedges under finite unions and complements. The result is our algebra.

Many constructions of rigid Boolean algebras have been given. In [vDMR] there is a short history. van Douwen, Monk and Rubin ask for a "natural" construction of a rigid Boolean algebra. We offer our construction as a candidate.

van Douwen, in [vD] has a similar construction. He begins with a tree T satisfying (a) and (c) and with the property that for all $\alpha \in T$, κ_α , the number of immediate successors to α , satisfies $\kappa_\alpha^{\aleph_0} = \kappa_\alpha$. He then topologizes the tree by taking certain infinite unions of wedges. The result is a rigid 0-dimensional compact space. The algebra of closed-and-open sets is the desired rigid Boolean algebra.

A Boolean algebra B is homogeneous if for any nonzero $b \in B$, the set $\{a \in B: a \leq b\}$ viewed as a Boolean algebra is isomorphic to B . It is weakly homogeneous if for all distinct nonzero $a, b \in B$ there exist nonzero $b_1 \leq b, a_1 \leq a$ such that the algebras consisting of $\{c: c \leq b_1\}$ and $\{c: c \leq a_1\}$ are isomorphic. We answer a question of Rubin in [R] by constructing a weakly homogeneous Boolean algebra that has no homogeneous factors.

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²This paper is dedicated to Merry Havens.

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In a subsequent paper we will present more general results concerning Boolean algebras obtained from trees in the fashion described here. We call this class of Boolean algebras tree algebras. See [B1 and B2] for more results involving tree algebras.

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1. Basic definitions and facts.

DEFINITION 1.1. (a) $\langle T, \leq \rangle$ is a *tree* iff $\leq$ partially orders T and $\forall \alpha \in T, \{\beta \in T: \beta \leq \alpha\}$ is well ordered by $\leq$. We will use T to denote both the set of nodes and the tree. α is a *root* of T iff α is minimal in T . For all $\alpha \in T$ we let A_α denote the set of $\leq$ -immediate successors to α . Two elements α, β are *comparable* iff $\alpha \leq \beta$ or $\beta \leq \alpha$. Otherwise they are *incomparable*. For all $\alpha \in T$, *level* α = order type $\{\beta: \beta < \alpha\}$. We define $\text{ht } T = \sup\{(\text{level } \alpha) + 1: \alpha \in T\}$.

The symbol $\leq$ will also be used to denote order in Boolean algebras. Thus for a, b in Boolean algebra B , $a \leq b$ iff $a \cdot -b = 0$. The exact meaning of $\leq$ will be clear from context. For any set L we let $\text{card } L$ denote the cardinality of L .

(b) For any tree T we define B_T , the *tree algebra* generated from T , as follows. For all $\alpha \in T$, let $S_\alpha = \{\beta \in T: \alpha \leq \beta\}$; then define B_T as the closure of $\{S_\alpha: \alpha \in T\}$ under finite unions and complements relative to the set T .

B_T is a Boolean algebra under the operations of set union, intersection and complement. In this paper when we use the word "algebra" we mean "Boolean algebra." We now give some easy technical results.

DEFINITION 1.2. (a) Given algebra B and $b \in B$, $B \upharpoonright b$ denotes the algebra whose universe is $\{a \in B: a \leq_b b\}$ with operations $^+ B \upharpoonright b, ^\circ B \upharpoonright b$ restrictions of the corresponding operations in B and $^- B \upharpoonright b$ defined thus: $^-_{B \upharpoonright b} a = b \cdot -_B a$.

(b) Given tree T and $\alpha \in T$, $T \upharpoonright \alpha$ denotes the tree whose universe is $\{\beta \in T: \alpha \leq \beta\}$ ordered by restricting the ordering on T .

LEMMA 1.3. Let T be a tree.

(a) $\forall b \in B_T \setminus \{0\}, b = \sum_{i \in n} f_i$ with n finite and $f_i \cdot f_j = 0$ for $i \neq j$ and each f_i of the form $f_i = S_{\alpha_i} - \sum_{\beta \in J_i} S_\beta$ (J_i a finite set of successors to α_i) unless T has more than one root in which case f_i may also have the form $f_i = -\sum_{\beta \in J} S_\beta$ (J finite).

(b) $\{S_\alpha: \alpha \in T\} \cup \{b \in B_T: b \text{ is an atom}\}$ is dense in B_T .

(c) If $\beta \in T$ then $B_T \upharpoonright S_\beta = B_{T \upharpoonright \beta}$.

PROOF. (a) $\{S_\alpha: \alpha \in T\}$ generates B_T so $b = \sum_{i \in n} \prod_{j \in J_i} S_{\alpha_{i,j}}^{\epsilon_{i,j}}$ where $\epsilon \in \{-1, 1\}$; $S_\alpha^{-1} = -S_\alpha$; $S_\alpha^1 = S_\alpha$; n, J_i finite; and for $i \neq i'$,

$$\prod_{j \in J_i} S_{\alpha_{i,j}}^{\epsilon_{i,j}} \cdot \prod_{j \in J_{i'}} S_{\alpha_{i',j}}^{\epsilon_{i',j}} = 0.$$

Each

$$\prod_{j \in J_i} S_{\alpha_{i,j}}^{\epsilon_{i,j}} = \prod_{j \in J_i'} S_{\alpha_{i,j}} \cdot \prod_{j \in J_i''} -S_{\alpha_{i,j}}$$

and $\forall i \in n$ either $\prod_{j \in J'_i} S_{\alpha_{i,j}} = S_{\alpha_{i,j}}$ for some $j \in J'_i$ or $\prod_{j \in J'_i} S_{\alpha_{i,j}} = 0$. The desired result now follows when we note that if α is the unique root of T and J is finite then $-\sum_{\beta \in J} S_\beta = S_\alpha - \sum_{\beta \in J} S_\beta$.

(b) The result follows immediately from (a) when we note that b is an atom iff for some $\alpha \in T$, A_α is finite and $b = S_\alpha - \sum_{\beta \in A_\alpha} S_\beta$.

(c) This is clear.

2. The construction.

DEFINITION 2.1. Let B be an algebra and $b \in B$. We say that $P \subset B$ is a *partition* of B (or of b) iff

1. $\sum P = 1_B$ (or b),
2. for all distinct $p, q \in P$, $p \cdot q = 0$ and
3. $0 \notin P$.

LEMMA 2.2. Let κ be an uncountable regular cardinal and T a tree satisfying

1. $\text{ht } T \leq \omega$,
2. $\forall \alpha \in T, \text{card } A_\alpha \neq \kappa$,
3. T has a single root

then B_T does not have a partition of cardinality κ .

PROOF. Suppose that T is a tree satisfying the hypotheses and $P = \{e_i: i \in \kappa\}$ is a partition of B_T with $e_i \neq e_j$ for $i \neq j$. Without loss of generality we may assume that each e_i is of the form $e_i = S_{\alpha_i} - \sum_{j \in J_i} S_{\beta_{i,j}}$ with J_i finite and $\alpha_i < \beta_{i,j}$ for all i, j . Let $H_P = \{\alpha_i: i \in \kappa\}$; then $\text{card } H_P = \kappa$ and $\alpha_i \neq \alpha_j$ for $i \neq j$ since $e_i \cdot e_j = 0$ for $i \neq j$. T has at most ω levels and $\text{cf } \kappa > \omega$ so κ -many elements of H_P occur on the same level.

Let r be the least element of ω such that there is a tree T satisfying

1. $\text{ht } T \leq \omega$,
2. $\forall \alpha \in T \text{ card } A_\alpha \neq \kappa$,
3. T has a single root,
4. B_T has a partition P , each element of the form $S_\alpha - \sum_{\beta \in J} S_\beta$ with J finite and $\alpha < \beta$ for all $\beta \in J$; and $\text{card } P = \kappa$,
5. H_P has κ -many elements on level r .

Since the trees that we are considering each have a single root, $r \neq 0$. Let $T, P, H = H_P$ witness our choice of r . Let α be the root of T . We will derive a contradiction.

Case 1. $\text{card } A_\alpha > \kappa$. We claim that $\alpha \in H$.

For suppose $\alpha \notin H$ then $\forall \delta \in H, S_\beta$ lies beneath exactly one element of $\{S_\gamma: \gamma \in A_\alpha\}$, say S_{γ_δ} . $\forall \delta \in H, S_\delta \leq S_{\gamma_\delta}$ so $\bigcup_{\delta \in H} S_\delta \subseteq \bigcup_{\delta \in H} S_{\gamma_\delta}$. But $\sum_{\delta \in H} S_\delta = 1$ so $\sum_{\delta \in H} S_{\gamma_\delta} = 1$. This is possible only if $\{\gamma_\delta: \delta \in H\} = A_\alpha$. Thus $\text{card } H \geq \text{card } A_\alpha > \kappa$, a contradiction. So $\alpha \in H$.

So we have that for some $i \in \kappa$ and finite $J, e_i = S_\alpha - \sum_{\beta \in J} S_\beta$. Note that $\forall \beta \in J, e_j \cdot S_\beta \neq 0 \Rightarrow e_j \leq S_\beta$ hence $\forall \beta \in J, S_\beta = \sum\{e_i: e_i \leq S_\beta\}$ so $\forall \beta \in J, \{e_i: e_i \leq S_\beta\}$ is a partition of S_β . Since J is finite, we can pick $\beta \in J$ so that

$$\text{card}\{e_i: \alpha_i \text{ is on level } r \text{ and } e_i \leq S_\beta\} = \kappa.$$

Now $\{e_i: e_i \leq S_\beta\}$ is a partition of $B \restriction S_\beta$, so by Lemma 1.3 (c) $P^* = \{e_i: e_i \leq S_\beta\}$ is a partition of $B_{T \restriction \beta}$ and $T \restriction \beta, P^*, H_{P^*}$ contradict the choice of r .

Case 2. $\text{card } A_\alpha < \kappa$. Then for some $\beta \in A_\alpha$, $\text{card}\{\gamma \in H: \gamma \text{ is on level } r \text{ and } \beta < \gamma\} = \kappa$. The argument used in Case 1 shows that $T \restriction \beta, P^*, H_{P^*}$ again contradict the choice of r .

DEFINITION 2.3. (a) B is a *rigid* algebra iff the identity is the only automorphism of B .

(b) B is a *homogeneous* algebra iff $\forall$ nonzero $b \in B$, $B \restriction b \cong B$. B is a *weakly homogeneous* algebra iff $\forall$ nonzero $a, b \in B$, $\exists$ nonzero $a_1 \leq a, b_1 \leq b$ such that $B \restriction a_1 \cong B \restriction b_1$.

THEOREM 2.4. Let T be a tree satisfying

1. $\text{ht } T = \omega$ and T has a single root.
2. $\forall \alpha, \beta \in T (\alpha \neq \beta \Rightarrow \text{card } A_\alpha \neq \text{card } B_\beta)$.
3. $\forall \alpha \in T$, $\text{card } A_\alpha$ is regular and infinite.

Then B_T is a rigid algebra.

PROOF. Claim 1. $\forall \alpha \in T$, $\forall$ uncountable, regular κ , S_α has a partition of size κ iff $\exists \beta \geq \alpha, \text{card } A_\beta = \kappa$. The implication from right to left is trivial. To go from left to right, suppose there is no $\beta \geq \alpha$ with $\text{card } A_\beta = \kappa$. Then $\kappa, T \restriction \alpha$ satisfy the hypotheses of Lemma 2.2 so $B_{T \restriction \alpha}$ has no partition of size κ . Thus $B \restriction S_\alpha$ has no partition of size κ .

Now suppose that φ is an automorphism of B .

Claim 2. $\forall \alpha \in T$, $\varphi(S_\alpha) \leq S_\alpha$. If the claim is false then there is some $b \in B_T$ such that $0 \neq b \leq \varphi(S_\alpha) \cdot -S_\alpha$. Since $\forall \alpha \in T$, A_α is infinite; by the remark in the proof of Lemma 1.3(b), B_T contains no atoms. So by 1.3(b), there is some $\gamma \in T$ such that $S_\gamma \leq b$. Thus $\varphi(S_\alpha)$ has a partition of size $\text{card } A_\gamma$. Since $S_\gamma \not\leq S_\alpha$, $\alpha \not\leq \gamma$, so by the first claim S_α has no partition of size $\text{card } A_\gamma$. This contradicts the fact that φ is an isomorphism and we have Claim 2.

Suppose $\exists \alpha \in T$, $\varphi(S_\alpha) \neq S_\alpha$. Then $\exists b \in B_T, 0 \neq b \leq S_\alpha \cdot -\varphi(S_\alpha)$. We pick $\gamma \in T$ such that $S_\gamma \leq b$. Then $\varphi(S_\gamma) \leq \varphi(S_\alpha)$. But $S_\gamma \leq -\varphi(S_\alpha)$ so $\varphi(S_\gamma) \leq S_\gamma \leq -\varphi(S_\alpha)$. This implies that $\varphi(S_\gamma) = 0$, a contradiction. This completes the proof.

THEOREM 2.5. There exists a weakly homogeneous tree algebra with no homogeneous factors.

PROOF. Let T be the tree of $\text{ht } \omega$ with one root satisfying $\forall \alpha \in T$ (level $\alpha = n \Rightarrow \text{card } A_\alpha = \omega_n$).

$B = B_T$ has no homogeneous factors: Suppose $B \cong C \times D$ then $C \cong B \restriction b$ for some $b \in B$. Pick $\alpha, \beta \in T$ such that $S_\alpha, S_\beta \leq b$ and level $\alpha < \text{level } \beta$. Let $n = \text{level } \alpha$. By Lemma 2.2, $B \restriction S_\alpha$ has a partition of size ω_n but $B \restriction S_\beta$ does not. Thus $B \restriction b$ is not homogeneous.

B_T is weakly homogeneous: Take any nonzero $a_1, b_1 \in B_T$. Find $\alpha, \beta \in T$ such that $S_\beta \leq a_1, S_\beta \leq b_1$ and level $\alpha = \text{level } \beta$. So $B \restriction S_\alpha \cong B \restriction S_\beta$.

M. Rubin asked in [R] whether a weakly homogeneous algebra without homogeneous factors exists and if such an algebra can be complete. Solovay [S] and,

independently, Koppelberg in [K] showed that no such complete algebra exists. Our construction suggests the following question.

Question. For all κ is there a κ^+ -complete weakly homogeneous algebra with no homogeneous factors? In particular, suppose for cardinal κ , T is a tree of ht ω satisfying $\forall \alpha \in T$ card A_α is regular and greater than κ and $\forall \alpha, \beta \in T$ ($\alpha \neq \beta \rightarrow$ card $A_\alpha \neq$ card A_β). Let B_T^* be the closure of $\{S_\alpha: \alpha \in T\}$ under κ -unions and complements. Is B_T^* a weakly homogeneous algebra without homogeneous factors?

ADDED IN PROOF. (1) J. D. Monk solved the question we ask by obtaining a canonical form for B_T^* . The following observation of S. Koppelberg, which is probably known, replaces a more complicated lemma of Monk.

LEMMA A. Let T be a set, X a collection of subsets of T , κ any infinite cardinal and A the closure of X under complements relative to T and under unions of $\leq \kappa$ elements. Then for each $a \in A$, there is $Y \in [X]^{\leq \kappa}$ and $F \subset Y_2$ such that $a = \bigcup_{f \in F} \bigcap_{t \in Y} t^{f_t}$.

Lemma A immediately yields:

LEMMA B. If κ is an infinite cardinal and T is a single rooted tree of height $\leq \omega$, then every nonzero $b \in B_T$ can be written in the form $b = \sum_{i < \lambda} f_i$, with (a) $\lambda \leq 2^\kappa$, (b) $f_i \cdot f_j = 0$ for $i \neq j$ and $f_i \neq 0$ for each i , and (c) $f_i = S_{J_i} - \sum_{u \in J_i} S_u$ for some J_i , a set of pairwise incomparable successors to t_i of power $\leq \kappa$.

Now using proofs almost identical to those used to show Lemma 2.2 and Theorems 2.4 and 2.5 Monk obtains, respectively, Lemma C and Theorems I and II.

LEMMA C. Let κ and μ be regular cardinals, $\kappa \geq \omega$ and $\mu \geq 2^\kappa$ and T be a tree satisfying (a) ht $T \leq \omega$, (b) $\forall t \in T$, card $A_t \neq \mu$, and (c) T has a single root, then B_T does not have a partition of cardinality μ .

THEOREM I. If κ is a regular infinite cardinal and T satisfies

- (a) ht $T = \omega$ and T has a single root,
- (b) $\forall s, t \in T$, $s \neq t \Rightarrow$ card $A_s \neq$ card A_t ,
- (c) $\forall s, t \in T$, card A_t is regular and card $A_t \geq 2^\kappa$, then B_T is rigid.

THEOREM II. If T is a single rooted tree of height ω such that $\forall t \in T$, level $t = n \Rightarrow$ card $A_t = (2^\kappa)^{+(n+1)}$, then B_T^* is weakly homogeneous and has no homogeneous factor.

(2) Subsequent to Monk, S. Todorćević constructed κ -complete weakly homogeneous BA's with no homogeneous factors in cardinalities lower than Monk's. For all regular $\aleph_0 \leq \kappa \leq \lambda$, Todorćević constructs 2^λ nonisomorphic κ -complete BA's of cardinality $\aleph^\kappa$. So, for instance, he has a σ -complete example of power $2^{\aleph_0}$. Monk's σ -complete algebras have power $\geq ((2^{\aleph_0})^{+\omega})^{\aleph_0} \geq (2^{\aleph_0})^{+\omega}$.

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