AN ALTERNATE VARIATIONAL PRINCIPLE FOR $\Delta u - u + |u|^{r-1} \operatorname{sgn} u = 0$

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ABSTRACT. An alternate variational principle for the equation in the title has been proposed by H. A. Levine. We analyse the relation between this principle and the Rayleigh quotient that has been used previously for the variational study of this problem in R^N . The main result is an existence theorem for $W^{1,2}(R^N)$ -solutions of the variational problem posed by Levine.

1. Introduction. This note is concerned with the following differential equation in R^N ($N \geq 2$),

$$(1.1) \quad \Delta u - u + |u|^{r-1} \operatorname{sgn} u = 0,$$

and more generally with the problem,

$$(1.2) \quad \nabla(|\nabla u|^{p-1} \operatorname{sgn} \nabla u) - |u|^{p-1} \operatorname{sgn} u + |u|^{r-1} \operatorname{sgn} u = 0,$$

in R^N . Here

$$\operatorname{sgn} \nabla u = |\nabla u|^{-1} \nabla u \quad \text{if } \nabla u \neq 0, \quad \operatorname{sgn} \nabla u = 0 \quad \text{otherwise.}$$

Variational studies of (1.1) have made use of the "Rayleigh quotient"

$$(1.3) \quad \left(\int (|\nabla u|^2 + u^2) dx \right)^{r/2} / \int |u|^r dx$$

(here $\int(\cdot) dx$ denotes integration over R^N), see e.g. [1]. In his abstract [3], H. A. Levine discussed the case $r = 5$, $N = 2$ of (1.1) and proposed for the variational study of this equation the quotient

$$(1.4) \quad \int |\nabla u|^2 dx \left(\int u^2 dx \right)^{1/2} / \left(\int u^6 dx \right)^{1/2};$$

note that the latter quotient is not only homogeneous in u , like (1.3), but also is invariant under change of scale in R^2 . Since the preparation of the original version of this note the details of Levine's work have appeared in [4].

In what follows we shall denote, when p , r , N are assumed to be given,

$$(1.5) \quad J_1(u) = \frac{(\int (|\nabla u|^p + |u|^p) dx)^{1/p}}{(\int |u|^r dx)^{1/r}};$$

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note that the infimum of $J_1(u)$ over $C_0^\infty(R^N) \setminus \{0\}$ is $K_{p,r}^{-1}$, where $K_{p,r}$ denotes the norm of the imbedding $W^{1,p}(R^N) \subset L^r(R^N)$. (Our notation for Sobolev spaces follows that of [2].) For the appropriate values of p, r, N we define

$$(1.6) \quad J_0(u) = \frac{(\int |\nabla u|^p dx)^{\alpha/p} (\int |u|^p dx)^{\beta/p}}{(\int |u|^r dx)^{1/r}},$$

where α and β are positive and are uniquely determined by the requirements that J_0 be homogeneous in u and invariant under change of scale in R^N (i.e. for any real positive scalars τ, λ , if $w(x) = \lambda u(\tau x)$ then $J_0(w) = J_0(u)$). Indeed one easily sees that if $p > 1$ then J_0 can be formed in accordance with these requirements if and only if

$$(1.7) \quad 0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{N},$$

and that α, β are given by

$$(1.8) \quad \alpha = \left(\frac{1}{p} - \frac{1}{r} \right) N, \quad \beta = 1 - \alpha.$$

The pair of positive numbers (p, r) will be said to be *admissible for dimension N* if $p > 1$ and (1.7) holds.

Here we analyse the relation between these two variational principles. The interesting feature of the quotient J_0 is that (assuming $p > 1$) it can be formed in accordance with the requirements set down above precisely when the variational problem

$$J_1(u) = \min, \quad u \in W^{1,p}(R^N),$$

has a solution. One outcome of this analysis is a new necessary condition for a "ground state" solution of (1.2), namely (1.11). Our result is the following

THEOREM. *Let (p, r) be admissible for dimension N . Then there exists a nonnegative radially symmetric function $u \in W^{1,p}(R^N) \setminus \{0\}$ which minimizes the quotient J_1 over $W^{1,p}(R^N) \setminus \{0\}$. If u is any minimizer of J_1 satisfying the normalization*

$$(1.9) \quad \int (|\nabla u|^p + |u|^p) dx = \int |u|^r dx$$

then u is a weak solution of (1.2), i.e.,

$$(1.10) \quad \int (|\nabla u|^{p-1} (\text{sgn } \nabla u) \cdot \nabla v + |u|^{p-1} (\text{sgn } u) v) dx = \int |u|^{r-1} (\text{sgn } u) v dx,$$

for all $v \in W^{1,p}(R^N)$.

A function $u \in W^{1,p}(R^N) \setminus \{0\}$ minimizes J_1 over $W^{1,p}(R^N) \setminus \{0\}$ if and only if it satisfies

$$(1.11) \quad \alpha \int |u|^p dx = \beta \int |\nabla u|^p dx,$$

and minimizes J_0 over the same set.

The author is grateful to the referee for having suggested some simplifications, both in proofs and in presentation.

2. Proof of the Theorem. We begin by proving the final assertion. For $u \in W^{1,p}(R^N) \setminus \{0\}$ let $a = \beta \int |\nabla u|^p dx$ and $b = \alpha \int |u|^p dx$. Then since $a^\alpha b^\beta \leq \alpha a + \beta b$, with equality if and only if $a = b$, we have that

$$J_0(u) \leq [\alpha^\alpha \beta^\beta]^{1/p} J_1(u)$$

with equality if and only if (1.11) holds. For an appropriate choice of λ , the function w , given by $w(x) = u(\lambda x)$ will satisfy (1.11) while $J_0(w) = J_0(u)$. The assertion follows immediately.

In what follows we use the notation

$$\|u\|_{1,p} = \left(\int (|\nabla u|^p + |u|^p) dx \right)^{1/p},$$

for the norm on $W^{1,p}(R^N)$; $V^p(R^N)$ denotes the subspace of $W^{1,p}(R^N)$ that consists of all radially symmetric functions.

LEMMA 2.1. *Let $1 < p < r < \infty$. There exist positive constants C, γ such that for $u \in V^p(R^N)$ and $\rho_0 > 0$*

$$(2.1) \quad \int_{|x| \geq \rho_0} |u|^r dx \leq C \rho_0^{-\gamma} \|u\|_{1,p}^r.$$

PROOF. For u as indicated we have, by Hölder's inequality,

$$\begin{aligned} |u(x)|^p &\leq p \int_{|x|}^{\infty} |u|^{p-1} \left| \frac{\partial u}{\partial \rho} \right| d\rho \\ &\leq \left(\frac{p}{q} \right)^{1/q} \left(q \int_{|x|}^{\infty} |u|^p d\rho \right)^{1/q} \left(p \int_{|x|}^{\infty} \left| \frac{\partial u}{\partial \rho} \right|^p d\rho \right)^{1/p} \\ &\leq \left(\frac{p}{q} \right)^{1/p} \int_{|x|}^{\infty} \left(\left| \frac{\partial u}{\partial \rho} \right|^p + |u|^p \right) d\rho \end{aligned}$$

and hence

$$(2.2) \quad |u(x)|^p \leq \text{const } |x|^{-(N-1)} \|u\|_{1,p}^p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If

$$(2.3) \quad r > pN/(N-1),$$

then (2.1), with $\gamma = (N-1)r/p - N$, follows directly from (2.2). If (2.3) fails to hold then we choose $r_0 > pN/(N-1)$ and use the interpolation

$$\int_{|x| \geq \rho_0} |u|^r dx \leq \left(\int |u|^p dx \right)^{(r_0-r)/(r_0-p)} \left(\int_{|x| \geq \rho_0} |u|^{r_0} dx \right)^{(r-p)/(r_0-p)}$$

to obtain the asserted inequality.

REMARK. The inequality (2.1) and its significance in regard to the problem at hand were suggested by Nehari's paper [5].

LEMMA 2.2. Let the pair (p, r) be admissible for dimension N , then the space $V^p(R^N)$ imbeds compactly in $L^r(R^N)$.

PROOF. Since $V^p(R^N)$ is reflexive, it suffices to show that a sequence $\{w_n\}$ that converges weakly to 0 in $V^p(R^N)$ converges to 0 in norm in $L^p(R^N)$. This follows immediately from Lemma 2.1 and the compactness of the imbedding

$$W^{1,p}(B_\rho) \subseteq L^r(B_\rho)$$

where $B_\rho = \{x \in R^N : |x| \leq \rho\}$; cf. (1.7).

COMPLETION OF THE PROOF. Since both $W^{1,p}(R^N)$ and $L(R^N)$ are lattices in which u and $|u|$ have the same norm, replacement of u by $|u|$ does not affect the value of J_1 . If the nonnegative function $u \in W^{1,p}(R^N) \setminus \{0\}$ is replaced by its Schwarz symmetrization $\tilde{u} \in V^p(R^N)$ (see [6] and the references there) then $\int |u|^\sigma dx = \int |\tilde{u}|^\sigma dx$ for $1 \leq \sigma < \infty$ while $\int |\nabla \tilde{u}|^p dx < \int |\nabla u|^p dx$ unless $\tilde{u}$ is merely a translate of u . It follows that we can choose a minimizing sequence $\{u_n\}$ for J_1 in such a way that u_n is nonnegative and $u_n \in V^p(R^N) \setminus \{0\}$ for $n = 1, 2, \dots$.

$$(2.4) \quad \int |u_n|^r dx = 1, \quad n = 1, 2, \dots;$$

and $\{u_n\}$ is weakly convergent, say to u_0 , in $V^p(R^N)$. It follows from Lemma 2.2 that $\{u_n\}$ converges strongly to u_0 in $L^r(R^N)$. Thus in view of (2.4)

$$\int |u_0|^r dx = 1$$

and in particular $u_0 \neq 0$. In view of the weak lower semicontinuity of $\|\cdot\|_{1,p}$ on $W^{1,p}(R^N)$ it readily follows that u_0 minimizes J_1 over $W^{1,p}(R^N) \setminus \{0\}$. Let u denote the function that results when u_0 is normalized to satisfy (1.9). It follows by standard results in the calculus of variations that if u minimizes J_1 and satisfies (1.9) then u must be a weak solution of (1.1).

3. The case $\alpha = 1$. In the limiting case in which the second inequality in (1.7) is replaced by equality we have obviously

$$J_1(u) > J_0(u), \quad u \in W^{1,p}(R^N) \setminus \{0\},$$

while if $u_\lambda(x) = u(\lambda x)$ then $J_0(u_\lambda)/J_1(u_\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$. It follows that J_1 cannot assume a minimum on $W^{1,p}(R^N)$. The infimum of J_0 however can be explicitly calculated in this case, [6]. For example, in the case $p = 2$, the change of variable $t = \ln \rho$, $w(t) = \rho^{2/(r-2)}u(\rho)$, transforms the equation

$$\frac{d^2 u}{d\rho^2} + \frac{N-1}{\rho} \frac{du}{d\rho} + |u|^{r-1} \operatorname{sgn} u = 0,$$

to the autonomous equation

$$\ddot{w} + \left(N - \frac{2r}{r-2}\right) \dot{w} - \frac{2}{r-2} \left(N - \frac{2r-2}{r-2}\right) w + |w|^{r-1} \operatorname{sgn} w = 0.$$

When $\alpha = 1$, i.e. $N = 2r/(r - 2)$ then this reduces to

$$\ddot{w} + \left(\frac{2}{r-2} \right)^2 w + |w|^{r-1} \operatorname{sgn} w = 0,$$

which can be integrated. From the formulas found in [6] one sees that the minimizer of J_0 belongs to $W^{1,p}(R^N)$ only when $N > p^2$.

REFERENCES

1. C. V. Coffman, *Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions*, Arch. Rational Mech. Anal. **46** (1972), 81-95.
2. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, New York, 1969.
3. H. A. Levine, *A numerical estimate for the best constant in a two-dimensional Sobolev inequality with three integral norms*, Notices Amer. Math. Soc. **26** (1979), Abstract 79T-B16.
4. ———, *An estimate for the best constant in a Sobolev inequality involving three integral norms*, Ann. Math. Pura Appl. **124** (1980), 181-197.
5. Z. Nehari, *On a nonlinear differential equation arising in nuclear physics*, Proc. Roy. Irish Acad. **62** (1963), 117-135.
6. G. Talenti, *Best constant in Sobolev inequality*, Ann. Math. Pura Appl. **110** (1976), 353-372.

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