CONTRACTIONS WITHOUT CYCLIC VECTORS¹

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ABSTRACT. It is proved that if T is a completely nonunitary contraction on Hilbert space such that T^{*n} does not converge strongly to 0 as $n \rightarrow \infty$, there is an integer $N > 0$ so that none of the powers T^{*m} with $m \geq N$ has a cyclic vector. Both conditions on T are essential, and the integer N is not universal, i.e., it depends on T .

In this note we are concerned with contractions T of Hilbert space $\mathcal{H}$. In previous papers [3] we considered conditions which assure the existence of cyclic vectors for T (or T^*). Here we shall deal with a converse question, namely, to find contractions *without* cyclic vectors.

1. First let us recall [3 or 4, Chapter II] that if U and U_+ are the minimal unitary and isometric dilations of T , acting on the spaces $\mathcal{H}$ and $\mathcal{H}_+$, respectively ($\mathcal{H} \subset \mathcal{H}_+ \subset \mathcal{H}$), and if $\mathcal{R}$ is the subspace of $\mathcal{H}_+$ on which U_+ is unitary, i.e., the subspace

$$(1) \quad \mathcal{R} = \bigcap_{n=0}^{\infty} U_+^n \mathcal{H}_+,$$

then $\mathcal{R}$ reduces both U_+ and U to the unitary operator $R = U_+|_{\mathcal{R}}$. Moreover, setting $X = P_{\mathcal{R}}|_{\mathcal{H}}$ and $\mathcal{X} = \overline{P_{\mathcal{R}}\mathcal{H}}$ (where $P_{\mathcal{R}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}$), we have

$$(2) \quad XT^* = R^*X$$

and $R^*\mathcal{X} = (R^*X\mathcal{H})^\perp = (XT^*\mathcal{H})^\perp \subset (X\mathcal{H})^\perp = \mathcal{X}$; and hence,

$$(3) \quad \mathcal{X} \subset R^*\mathcal{X}.$$

Because of the minimality of U_+ , and since $\mathcal{R}$ reduces U_+ to R , we also have

$$(4) \quad \mathcal{R} = P_{\mathcal{R}}\mathcal{H}_+ = P_{\mathcal{R}} \bigvee_{k=0}^{\infty} U_+^k \mathcal{H} = \bigvee_{k=0}^{\infty} R^k P_{\mathcal{R}} \mathcal{H} = \bigvee_{k=0}^{\infty} R^k \mathcal{X}.$$

Taking (3) into account we infer that the sequence $\{R^k \mathcal{X}\}_{k=0}^{\infty}$ of subspaces of $\mathcal{H}_+$ is increasing so the last member of (4) is equal to $\bigvee_{j=1}^{\infty} R^{k_j} \mathcal{X}$ for any sequence of integers $k_j \rightarrow \infty$ (as $j \rightarrow \infty$); i.e.,

$$(5) \quad \mathcal{R} = \bigvee_{j=1}^{\infty} R^{k_j} \mathcal{X}.$$

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2. Let $\{E_t\}_{0 \leq t < 2\pi}$ be the spectral family for R , i.e., $R = \int_0^{2\pi} e^{it} dE_t$, and let $\{E(A)\}$ be the spectral measure associated with $\{E_t\}$, where A runs over all Borel subsets of $[0, 2\pi)$; i.e., $E(A) = \int_0^{2\pi} \chi_A(t) dE_t$.

Denoting by $|A|$ the Lebesgue measure of A , and by C complementation with respect to $[0, 2\pi)$, let us define

$$(6) \quad \alpha_T = \inf\{|A| : E(CA) = 0\}.$$

LEMMA. (a) The infimum α_T is attained for some $A = A_\infty$ with $E(CA_\infty) = 0$.

(b) If the contraction T is completely nonunitary and T^{*n} does not tend to 0 (strongly) as $n \rightarrow \infty$, then $\alpha_T > 0$.

PROOF. Choose a minimizing sequence $\{A_n\}_1^\infty$ of Borel subsets of $[0, 2\pi)$, i.e. such that $E(CA_n) = 0$ and $|A_n| \rightarrow \alpha_T$. Then the set

$$A_\infty = \bigcap_1^\infty A_n = C \left(\bigcup_1^\infty CA_n \right)$$

obviously satisfies the requirements of (a).

If the contraction T is completely nonunitary, then Theorem II.6.4 of [3] tells us that the minimal unitary dilation U of T has spectral measure absolutely continuous with respect to Lebesgue measure; and this property of U is obviously inherited by $R = U|_{\mathfrak{H}}$. Thus $|A| = 0$ implies $E(A) = 0$. In particular, if $\alpha_T = |A_\infty| = 0$, then $E(A_\infty) = 0$. Since, on the other hand, $E(CA_\infty) = 0$, this leads to the conclusion $E[0, 2\pi) = 0$ i.e., $\mathfrak{H} = \{0\}$. This in turn contradicts, by virtue of Theorem II.2.1 of [4], the assumption $T^{*n} \not\rightarrow 0$. Thus part (b) of the Lemma is also proven.

3. THEOREM. Let T be a completely nonunitary contraction on $\mathfrak{H}$ for which T^{*n} does not tend to 0 (strongly) as $n \rightarrow \infty$. Then for all m large enough, namely for $m > 2\pi/\alpha_T$, T^{*m} has no cyclic vector in $\mathfrak{H}$.

PROOF. Let us assume that, contrary to what was asserted, there exists a cyclic vector h for T^{*m} , i.e., such that $\bigvee_{n=0}^\infty (T^{*m})^n h = \mathfrak{H}$. Then we have for $r = Xh$, taking into account also the fact that (1) implies $XT^{*n} = R^{*n}X$ ($n = 0, 1, 2, \dots$),

$$(7) \quad \bigvee_{k=0}^\infty (R^{*m})^k r = \bigvee_{k=0}^\infty X(T^{*m})^k h = (X\mathfrak{H})^\perp = \mathfrak{H}.$$

On the other hand, applying (5) with the sequence of exponents mj ($j = 0, 1, \dots$) we have, by (7),

$$(8) \quad \mathfrak{H} = \bigvee_{j=0}^\infty R^{mj} \mathfrak{H} = \bigvee_{j,k=0}^\infty R^{m(j+k)} r = \bigvee_{n=-\infty}^\infty (R^m)^n r.$$

Now consider any Borel set A in $[0, 2\pi)$ for which $(E(A)r, r) = (r, r)$, i.e., $E(CA)r = 0$. Since R commutes with its spectral measure we have $E(CA)q(A)r = q(A)E(CA)r = 0$ for every trigonometric polynomial $q(R)$ of R . Since by virtue of (8) every $r' \in \mathfrak{H}$ is the (strong) limit of some sequence $\{q_n(R)r\}_n$, we have $E(CA)r' = 0$ for every $r' \in \mathfrak{H}$; and thus $E(CA) = 0$, $E(A) = I_{\mathfrak{H}}$. Therefore, $|A| \geq \alpha_T$.

Moreover, since $s = (\int_0^{2\pi} t dE_t)r \in \mathfrak{H}$, it follows from (8) that there exists a sequence $\{p_n(z)\}_{n=1}^\infty$ ($z = e^{it}$) of trigonometric polynomials such that $p_n(R^m)r \rightarrow s$, that is,

$$\int_0^{2\pi} |p_n(e^{imt}) - t|^2 d(E_t r, r) = \|p_n(R^m)r - s\|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Replacing the sequence $\{p_n\}$, if necessary, by an appropriate subsequence (still denoted by $\{p_n\}$) we have even pointwise convergence

$$(9) \quad p_n(e^{imt}) \rightarrow t, \quad \text{as } n \rightarrow \infty,$$

almost everywhere with respect to the measure $(E(\cdot)r, r)$, i.e., on a (necessarily Borel) set A_m of full $(E(\cdot)r, r)$ -measure: $(E(A_m)r, r) = (r, r)$. As it was shown above this implies

$$(10) \quad |A_m| \geq \alpha_T.$$

As the functions $p_n(e^{imt})$ are periodic in t , with period $2\pi/m$, relation (9) implies that A_m has the property

$$(11) \quad \text{if } t_1, t_2 \in A_m \text{ are such that } t_1 - t_2 = 2\pi k/m \text{ for some integer } k, \text{ then } t_1 = t_2.$$

Consider the sets

$$B_k = A_m \cap \left[\frac{2\pi}{m}k, \frac{2\pi}{m}(k+1) \right) \quad (k = 0, 1, 2, \dots, m-1)$$

and their copies $B'_k = B_k - 2\pi k/m$ shifted into $[0, 2\pi/m)$. From (11) we get that the sets B'_k are mutually disjoint. Hence,

$$\frac{2\pi}{m} \geq \left| \sum_{k=0}^{m-1} B'_k \right| = \sum_{k=0}^{m-1} |B'_k| = \sum_{k=0}^{m-1} |B_k| = |A_m|.$$

In view of (10) this contradicts the choice of m in the Theorem.

This completes the proof.

4. Remarks. (a) The Theorem does not hold if the assumption that T be completely nonunitary is dropped.

Indeed, consider the operator T on 2-space $\mathfrak{H}_2$ given by the matrix $\text{diag}[1, \varepsilon]$, $\varepsilon = e^{2\pi i a}$, where a is a real, irrational number. T is unitary so T^{**n} does not converge (strongly) to 0 as $n \rightarrow \infty$. But T^{*m} has, no matter how large m is, a cyclic vector. Moreover, every vector $h = \begin{bmatrix} a \\ b \end{bmatrix}$ with $a, b \neq 0$ is cyclic for T^{*m} ($m \geq 1$); indeed h and $T^{*m}h$ already span $\mathfrak{H}$: for every $k = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathfrak{H}$ we have $c_0 h + c_1 T^{*m}h = k$ for some numbers c_0, c_1 ; this is a consequence of the fact that $\varepsilon^{-m} \neq 1$.

(b) On the other hand, the mere condition that T be completely nonunitary does not suffice. Indeed, the simple unilateral shift S on l^2 is clearly completely nonunitary, but, for every fixed $m > 2$, S^{*m} , being unitarily equivalent to $S^* \oplus S^* \oplus \dots \oplus S^*$ (m terms), has a cyclic vector. (See [2, Problem 126].)

(c) The Theorem cannot be improved so that a given power, say T^{*N} , is not cyclic for all T considered.

To prove this, consider the function

$$\theta(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log k_N(t) dt\right) \quad (|z| < 1),$$

$$k_N(t) = \begin{cases} 1 & \text{on } (2\pi/N, 2\pi), \\ 1/2 & \text{on } [0, 2\pi/N]. \end{cases}$$

This function belongs to the Hardy class H^∞ , and the radial limit $\theta(e^{it}) = \lim_{r \rightarrow 1} \theta(re^{it})$ exists and satisfies $|\theta(e^{it})| = k_N(t)$ almost everywhere on $0 \leq t \leq 2\pi$, with respect to Lebesgue measure. Hence,

$$1 - |\theta(e^{it})|^2 = 1 - |k_N(t)|^2 = \begin{cases} 0 & \text{on } (2\pi/N, 2\pi), \text{ a.e.}, \\ 3/4 & \text{on } [0, 2\pi/N], \text{ a.e.} \end{cases}$$

In particular, $\theta(z)$ is a purely contractive analytic function ($\theta(0) = 2^{-N/2}$); thus it coincides with the characteristic function of a contraction T [4, Chapter VI], which is unitarily equivalent to the operator V_N of multiplication by e^{it} on the space $L^2(0, 2\pi/N)$ (Theorem V1.6.1 in [4]). Therefore, T^{*N} is unitarily equivalent to the operator of multiplication by e^{-it} , and so also to the operator of multiplication by e^{it} , on $L^2(0, 2\pi)$. But this operator is cyclic (with the cyclic vector $e^{-1/t} \in L^2(0, 2\pi)$, say). Thus T^{*N} also has a cyclic vector.

(d) Our Theorem is an analogue of a deep result of Berger [1] concerning hyponormal operators, and was partly inspired by some comments on this result made by R. G. Douglas and C. Pearcy.

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