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ONE-TO-ONE OPERATORS ON FUNCTION SPACES

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ABSTRACT. For a Banach algebra A one-to-one operators with closed range on $C_0(S, A)$ are characterized in terms of the associated vector measures given by the Riesz Representation Theorems. Multiplicatively symmetric operators are also studied.

1. Introduction. Let S be a locally compact Hausdorff topological space and let A, B be Banach algebras. Denote by $C_0(S, A)$ the algebra of continuous functions from S to A vanishing at infinity endowed with the uniform norm. If $A = \mathbf{C}$, the set of all complex numbers, we simply write $C_0(S)$. Let $C_0(S) \hat{\otimes} A$ be the completion of the algebraic tensor product $C_0(S) \otimes A$ with respect to the least cross norm. Then $C_0(S, A) = C_0(S) \hat{\otimes} A$, where the equation indicates isometry between the two spaces.

Continuous linear operators $T: C_0(S, A) \to B$ are represented by measures $m: \mathfrak{B}(S) \to \mathfrak{L}[A, B^{**}]$ in, for example, Brooks and Lewis [4] and Batt and Berg [3]; where $\mathfrak{B}(S)$ is the σ-algebra of Borel subsets of S and B^{**} is the second dual of B. Mapping properties of representing measures are studied, for example, by Bilyeu and Lewis [2], Brooks and Lewis [4], Johnson [9] and the author [5].

A bounded linear operator $T: C_0(S, A) \to C_0(S, A)$ is called a *multiplicatively symmetric operator* if $T(fT(g)) = T(T(f)g)$ for all $f, g \in C_0(S, A)$. Dhombres [6] showed that multiplicatively symmetric operators coincide with *exaves* for the case when S is compact, $A = \mathbf{C}$ and $\|T\| = T(1) = 1$. In §2 we characterize one-to-one operators with closed range and multiplicatively symmetric operators which are one-to-one in terms of their representing measures. An example is given in §3 to explain Theorem 2.1. More interesting examples of representing measures may be found, for example, in Brooks and Lewis [4].

Throughout this paper multiplication in the second dual of Banach algebras is defined by the left Arens product. Duncan and Hosseiniun [8] is a convenient reference for the Arens product.

2. One-to-one operators. For $f' \in C_0^*(S, A)$, $x \in A$, there is a unique regular Borel measure $\mu(x, f')$ such that $\int f \, d\mu(x, f') = f'(f \cdot x)$ for $f \in C_0(S)$. Therefore, for $e \in \mathfrak{B}(S)$, $x \in A$, $1_e \otimes x$ can be viewed as an element of $C_0^{**}(S, A)$ defined by $(1_e \otimes x)(f') = \mu(x, f')(e)$. Recall that $T^{**}(1_e \otimes x) = m(e)x$. Let $\mathfrak{P}(S)$ be the class of all Borel partitions of S.

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THEOREM 2.1. *An operator* $T: C_0(S, A) \to C_0(S, A)$ *is one-to-one and has a closed range iff* $\sum_{i=1}^n m(e_i)x_i = 0$ *implies that* $x_i = 0$ *for* $\{e_i\} \in \mathfrak{P}(S)$ *and* $x_i \in A$ *with* $\|x_i\| \leq 1$ $(i = 1, 2, \ldots, n)$.

PROOF. Suppose, for every $\sum_{i=1}^n (1_{e_i} \otimes x_i)$, $T^{**}(\sum 1_{e_i} \otimes x_i) = 0$ implies $\sum(1_{e_i} \otimes x_i) = 0$. Let g_0' be any element in $C_0^*(S, A)$ and let

$$M = \left\{T^{**}(\textstyle\sum 1_{e_i} \otimes x_i) : \{e_i\} \in \mathfrak{P}(S), x_i \in A, \|x_i\| \leq 1\right\}.$$

Then M is a linear subspace of $C_0^{**}(S, A)$. Define a linear functional f_0' on M by

$$f_0'\big(T^{**}(\textstyle\sum 1_{e_i} \otimes x_i)\big) = \big(\textstyle\sum 1_{e_i} \otimes x_i\big)(g_0').$$

Since $T^{**}(\sum 1_{e_i} \otimes x_i) = 0$ implies $\sum 1_{e_i} \otimes x_i = 0$, f_0' is well defined. Furthermore,

$$\left|f_0'\big(T^{**}(\textstyle\sum 1_{e_i} \otimes x_i)\big)\right| = \left|\big(\textstyle\sum 1_{e_i} \otimes x_i\big)(g_0')\right| \leq \|g_0'\|.$$

Therefore f_0' is continuous on M and so can be extendedto a continuous linear functional f' on $C_0^{**}(S, A)$. Since

$$f'\big(T^{**}(\textstyle\sum 1_{e_i} \otimes x_i)\big) = \big(\textstyle\sum 1_{e_i} \otimes x_i\big)(g_0'),$$

we see, by taking the limit process, that

$$f'(T^{**}(f)) = f(g_0') \qquad (f \in C_0(S, A)).$$

Recall that $T^{**}(f) \in C_0(S, A)$ for all $f \in C_0(S, A)$. Hence, when f' is considered as a functional on $C_0(S, A)$, $g_0' = T^*f'$ and so T^* is an onto mapping. Therefore T is a one-to-one operator with closed range [**10**, Theorem 4.14].

Conversely, suppose T is one-to-one with closed range and $\sum m(e_i)x_i = 0$. Then T^* is onto [**10**, Corollary of 4.12] and, for all $f' \in C_0^*(S, A)$, $T^{**}(\sum 1_{e_i} \otimes x_i)(f') = 0$. Let $g' \in C_0^*(S, A)$. Then there is $f' \in C_0^*(S, A)$ such that $g' = T^*f'$. Hence

$$\big(\textstyle\sum 1_{e_i} \otimes x_i\big)(g') = \big(\textstyle\sum 1_{e_i} \otimes x_i\big)(T^*f') = T^{**}\big(\textstyle\sum 1_{e_i} \otimes x_i\big)(f') = 0.$$

That is $\sum(1_{e_i} \otimes x_i) = 0$ in $C_0^{**}(S, A)$ and so $x_i = 0$ for $i = 1, \ldots, n$ [**4**, Lemma 2.1].

It is easy to verify by an argument similar to [**5**, Proposition 2.2] that T: $C_0(S, A) \to C_0(S, A)$ is a multiplier iff T^{**}: $C_0^{**}(S, A) \to C_0^{**}(S, A)$ is a multiplier. Therefore T is a multiplier iff

$$\big(1_{e_1} \otimes x\big)\big(m(e_2)y\big) = \big(m(e_1)x\big)\big(1_{e_2} \otimes y\big)$$

for all $e_1, e_2 \in \mathfrak{B}(S)$, $x, y \in A$. The following theorem shows the difference between multipliers and multiplicatively symmetric operators. Brooks and Lewis defined supports, supp m, of weakly regular measures m and showed that supp m = supp T in [**4**].

THEOREM 2.2. *A one-to-one operator T is multiplicatively symmetric iff*

$$\big(1_{e_1} \otimes x\big)\big(m(e_2)y\big) = \big(m(e_1)x\big)\big(1_{e_2} \otimes y\big)$$

for all $e_1, e_2 \in \mathfrak{B}(\operatorname{supp} m)$ *and* $x, y \in A$.

PROOF. Suppose T is a one-to-one multiplicatively symmetric operator. Then

$$T(fT(g)) = T(T(f)g) \quad (f, g \in C_0(S, A)).$$

Therefore $fT(g) = T(f)g$ for all $f, g \in C_0(S, A)$. Hence, by an argument similar to [5, Proposition 2.1], $FT''(G) = T''(F)G$ for all $F, G \in C_0^{**}(S, A)$. In particular

$$(1_{e_1} \otimes x)(m(e_2)y) = (m(e_1)x)(1_{e_2} \otimes y)$$

for all $e_1, e_2 \in \mathfrak{B}(\operatorname{supp} m)$ and $x, y \in A$.

Conversely, suppose $(1_{e_1} \otimes x)(m(e_2)y) = (m(e_1)x)(1_{e_2} \otimes y)$ for $e_1, e_2 \in \mathfrak{B}(\operatorname{supp} m)$ and $x, y \in A$. Then

$$\left(\sum_{i=1}^{n} 1_{e_i} \otimes x_i\right)(m(e_j)y_j) = \left(\sum_{i=1}^{n} m(e_i)x_i\right)(1_{e_j} \otimes y_j)$$

for $e_i, e_j \in \mathfrak{B}(\operatorname{supp} m)$, $x_i, x_j \in A$. Since $\Sigma 1_{e_i} \otimes x_i \in C_0^{**}(S, A)$ for $e_i \in \mathfrak{B}(\operatorname{supp} m)$ and $x_i \in A$, each $f \in C_0(\operatorname{supp} m, A)$ can be considered as an element in $C_0^{**}(S, A)$. Using Bartle's Bounded Convergence Theorem, we see

$$f(m(e_j)y_j) = T^{**}(f)(1_{e_j} \otimes y_j)$$

and similarly $f(T^{**}(g)) = T^{**}(f)g$ for $f, g \in C_0(\operatorname{supp} m, A)$. For each $f \in C_0(S, A)$, the restriction of f to $\operatorname{supp} m$, f_m, is in $C_0(\operatorname{supp} m, A)$. Since $T(f) = \int f_m \, dm = T^{**}(f_m)$, we have $T^{**}(f_m) \in C_0(S, A)$ for every $f \in C_0(S, A)$. Therefore for f, $g \in C_0(S, A)$,

$$\begin{aligned} T(fT(g)) &= \int fT(G)\,dm = \int f_m T^{**}(g_m)\,dm \\ &= \int T^{**}(f_m)g_m\,dm = \int T(f)g\,dm \\ &= T(T(f)g). \end{aligned}$$

This completes the proof of the theorem.

3. Remarks. Theorem 2.1 is valid when A is only a Banach space. In the proof of sufficiency of Theorem 2.2, T is not necessarily to be one-to-one. It is interesting to determine whether Theorem 2.2 is true in general even if T is not one-to-one.

4. Example. The paper concludes with an example to illustrate Theorem 2.1. Let S be the natural numbers equipped with the discrete topology, and let A be a Banach space. Denote

$$1_1(A) = \left\{ x = (x_i) \colon x_i \in A \text{ and } \sum_{n=1}^{\infty} \|x_i\| < \infty \right\}.$$

Then $1_1(A)$ is a Banach space with the norm defined by $\|x\| = \Sigma\|x_i\|$ for $x = (x_i)$. It is shown by Dobrakov [7] that $C_0^*(S, A) = 1_1(A^*)$.

EXAMPLE 4.1. *Let H be a Hilbert space, $\{\alpha_n\}$ be a sequence of complex numbers converging to zero and* $|\alpha_n| \leq 1$. Define $T\colon C_0(S, H) \to C_0(S, H)$ by

$$Tf = (\alpha_i f_i) \quad (f = (f_i) \in C_0(S, H)).$$

Then the representing measure m of T is defined by $m(E) = \Sigma_{n \in E}\alpha_n e^n$, where $e^n = (e_i^n)$ with $e_i^n = 0$ if $i \neq n$ and $e_n^n = 1$. Then $m\colon \mathfrak{B}(S) \to \mathfrak{L}[H, C_0(S, H)]$ and

$m(E)x = (\sum_{n\in E}\alpha_n e^n)x$. Set

$$K = \left\{ \sum_{i=1}^{n} m(E_i)x_i \colon \{E_i\} \in \mathcal{P}(S),\, x_i \in H \text{ and } \|x_i\| \leq 1 \right\}.$$

Then $K \subset C_0(S, H)$. We shall show that K is weakly conditionally compact. Recall that a set in a Hilbert space is relatively weakly sequentially compact iff it is bounded. Let $\{\gamma^n\} = \{\sum_{i=1}^{N_n} m(E_i^n)x_i^n\}$ be in K. Then each γ^n is of the form $\gamma^n = (\alpha_i x_i^n)$. For each i, $\{\alpha_i x_i^n \colon n = 1, 2, \ldots\}$ is bounded in H and so, without loss of generality, we can assume that $\{\alpha_i x_i^n\} \to y_i \in H$ weakly. Hence, for each $f_i \in H$, $\langle \alpha_i x_i^n, f_i \rangle \to \langle y_i, f_i \rangle$. For each $F = (f_i) \in l_1(H) = C_0^*(S, H)$, since

$$\left| \sum_{i=1}^{\infty} \langle \alpha_i x_i^n, f_i \rangle - \sum_{i=1}^{N} \langle \alpha_i x_i^n, f_i \rangle \right| \leq \sum_{i=N+1}^{\infty} \|f_i\|,$$

it is easily verified that

$$F((\alpha_i x_i^n)) = \sum_{i=1}^{\infty} \langle \alpha_i x_i^n, f_i \rangle \to \sum_{i=1}^{\infty} \langle y_i, f_i \rangle = F((y_i)).$$

That is K is weakly conditionally compact and we conclude that T is weakly compact. Furthermore T is one-to-one and onto iff $\alpha_n \neq 0$ for each n.

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